

Finite Automata with Time-Delay Blocks ^{*}

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Abstract. The notion of delays arises naturally in many computational models, such as, in the design of circuits, control systems, and dataflow languages. In this work, we introduce *automata with delay blocks* (ADBs), extending finite state automata with variable time delay blocks, for deferring individual transition output symbols, in a discrete-time setting. We show that the ADB languages strictly subsume the regular languages, and are incomparable in expressive power to the context-free languages. We show that ADBs are closed under union, concatenation and Kleene star, and under intersection with regular languages, but not closed under complementation and intersection with other ADB languages. We show that the emptiness and the membership problems are decidable in polynomial time for ADBs, whereas the universality problem is undecidable. Finally we consider the linear-time model checking problem, i.e., whether the language of an ADB is contained in a regular language, and show that the model checking problem is PSPACE-complete.

1 Introduction

The class of dynamical systems (or processes) with delays occur frequently in control systems where delays arise due to physical constraints (see *e.g.* [CL07; GKC03; Sip+12; Zho06]). The notion of delays is also common in systems where transmission of information is involved. Delay blocks have been used for modeling such time delays in engineering systems, for example, the unit delay block in Simulink [Mat12b] delays the input signal by one sample period, corresponding to the z^{-1} discrete time Z -transform operator. The memory block in Simulink, meant for continuous time signals, delays the input by one integration time step. Mathworks' Control Systems Toolbox [Mat12a] can be used for modeling delays in control systems using the $e^{-\Delta s}$ Laplace transform operator (in the transfer functions) for modeling a delay of Δ time units; the coupling between the delay and the system dynamics is tracked in the internal state space model. The notion of delays arises naturally in other computational models, *e.g.*, time delays are used in the design and analysis of circuits (timing analysis and analysis of circuits with latches), and delays are a key component in dataflow languages (*e.g.* in the Ptolemy II framework [Eke+03; Pto]).

Although delay constructs have been widely used in control systems, design of circuits, and dataflow languages, they have not been considered in the classical automata theoretic settings in computer science. One approach to model delays in the automata theoretic setting has been by the introduction of an automaton model for an intermediate buffer for explicitly modeling the state of the buffer. This approach suffers from three crucial drawbacks: (1) the buffer length has to be fixed in any given model, (2) the buffer contents have to be explicitly modeled leading to unnecessary

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model complexity, and (3) the state space of the system blows up with increasing buffer size, due to state space modeling of the buffer contents.

In this work, we introduce an extension of the standard finite state automata model by enriching automata with variable discrete-time delay blocks for deferring individual output symbols. We call the resultant structures *automata with delay blocks* (ADBs). Viewing the automata as generators of strings, the string generated by an accepting run of a standard finite state automaton is the same as the sequence of symbols observed as the output of the run. In automata with delay blocks, the output symbols are *generated* by a regular automaton structure, but the *output sequence* of the symbols differs from the symbol generation sequence due to the delay blocks involved. In an ADB, there is an associated discrete-time delay Δ with each transition e labelled by an output symbol; in the output the symbol labeling the edge e appears after a delay of Δ time units. Time passes in the model in discrete time steps, either via an explicit *tick* transition in the ADB, or when the automaton run ends in an accepting state. We present a couple of examples to illustrate the model. Given an ADB \mathcal{A} , let $\mathcal{L}(\mathcal{A})$ denote the (discrete-time) output language of the automaton, and let $\mathcal{U}(\mathcal{A})$ denote the untimed output language.

Example 1. Consider a shipwreck scenario where hazardous material containers from a wrecked ship are floating in the ocean, and are being dispersed by ocean currents. A team of autonomous underwater vehicles (AUVs) is monitoring the situation, their goal being to (1) detect the possible locations of the drums using sonar data, and (2) monitor affects on underwater marine life due to leaking materials from the containers. For illustrative purposes, consider a team of two vehicles named AUV-1 and AUV-2. AUV-1 is operating at a depth of 10 meters, and is taking sonar imaging data above it and processing it to detect the floating drum locations. AUV-2 is operating at a depth of 150 meters and monitoring the underwater marine life situation. The search pattern of AUV-2 depends on the possible sightings of containers given by AUV-1 which are conveyed through acoustic communication. AUV-1 periodically, surfaces as it is close to the surface, sends its full detailed imaging data to the base station through GSM communication (high datarate and only works above water, underwater acoustic communication is extremely low datarate and has limited range) where human operators study data and update the earlier sighting inferences of AUV-1, and send the updates back to AUV-1, which must then convey the updates back to AUV-2 through underwater acoustic communication. The human operators may also change the resurfacing frequency of AUV-1 depending on the data received.

We are interested in describing the pattern of messages received by AUV-2. We define one discrete time unit to be the time in between two AUV-1 resurfacings (note that this corresponds to variable physical times, and a variable number of point monitorings). In one such time unit, AUV-1 sends k point locations to AUV-2, each annotated with y and m (for possible container sightings, m denotes “maybe”). The updates from the base station are conveyed to AUV-2 in the next time slot from AUV-1 as simply a k -bit sequence corresponding to the same k locations as in the previous time slot (the locations are not sent again to AUV-2 as underwater communication is extremely expensive). Let us denote the sending of the point coordinates as the event p . Then the (untimed) language describing the pattern of messages from AUV-1 to AUV-2 is

$$\{(w\#w') \mid w \in \{py, pm\}^* \text{ and } w' \in \{y, n\}^* \text{ and } 2|w'| = |w|\}^*$$

where $\#$ denotes the demarcation between two adjacent time slots. This language can be described in a natural and intuitive fashion by the ADB in Figure 1. The automaton also makes it clear that the i -th y, n that appears after the $\#$ corresponds to the i -th py, pm in the previous time slot; this

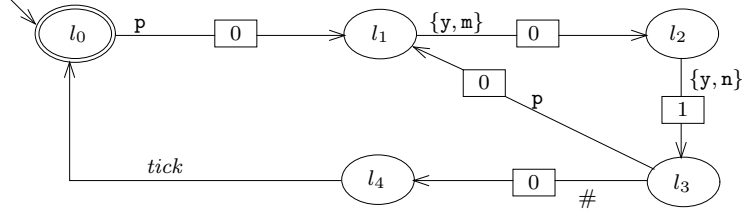


Fig. 1. Automaton \mathcal{A}_0 with delay blocks.

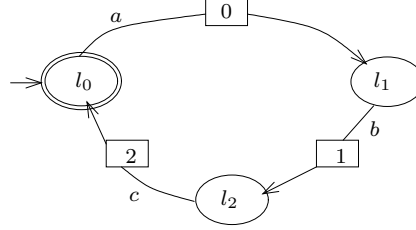


Fig. 2. Automaton \mathcal{A}_1 with delay blocks.

relationship may be useful for further processing of the point coordinates values. We explain the workings of the automaton \mathcal{A}_0 in detail below.

The initial location is l_0 which is also the only accepting location. Each edge has a delay block, with the number inside the block denoting the time delay associated with the block. Consider accepting runs of the automaton. The output symbols are *generated* in accepting runs according to the regular expression sequence $(p\{y, m\}\{y, n\})^*$ (the transition labeled *tick* denotes time passing by one time unit and *tick* is not an output symbol). However, because of the associated delays with the transitions, the output symbols (namely p, y, m, n) appear in a different sequence. Consider a particular run sequence $r = p y y \# \text{ tick } p y n p m n \# \text{ tick}$. Recall that time advances in ADBs either via the explicit *tick* transition, or when the run ends in an accepting state. Thus, in the run r , the first four symbols (*i.e.* $p y y \#$) are generated at time 0. The second y symbol has an associated delay of 1, the rest have an associated delay of 0. The 0-delay symbols appear immediately in the output (at time 0). Then, we have the first *tick* transition, which results in time advancing to 1. At time 1, first the 1-delay symbol, y (generated previously) appears at the output. Then, the sequence $p y n p m n \#$ is generated, with the first and the second n symbols having a delay of 1. Except for these two delayed n symbols, the rest appear immediately at time 1. Then comes the second *tick* transition which results in time advancing to 2, and at time 2, the two delayed n symbols appear. Thus, the time stamped output sequence corresponding to the run r after time 2 is $\langle p, 0 \rangle \langle y, 0 \rangle \langle \#, 0 \rangle \langle y, 1 \rangle \langle p, 1 \rangle \langle y, 1 \rangle \langle p, 1 \rangle \langle m, 1 \rangle \langle \#, 1 \rangle \langle n, 2 \rangle \langle n, 2 \rangle$ (the second element in the tuples denotes the timestamp when the first element of the tuple appears in the output). \square

Example 2. Consider the ADB \mathcal{A}_1 in Figure 2. The initial state is l_0 , which is also the only accepting state. Consider an accepting run of the automaton. The output symbols are generated in accepting runs according to the regular expression sequence $(abc)^*$. However, the output delay associated with the transition for a is 0, for b is 1, and for c the delay is 2. As there are no explicit time advancing *tick* transitions, time advances only when the run ends in the accepting state, and then the symbols with delay 0 are observed (according to their generation sequence), then the symbols at time 1, and so on. It can be seen that the output symbol sequence for the ADB \mathcal{A}_1 is $a^n b^n c^n$. Thus, the

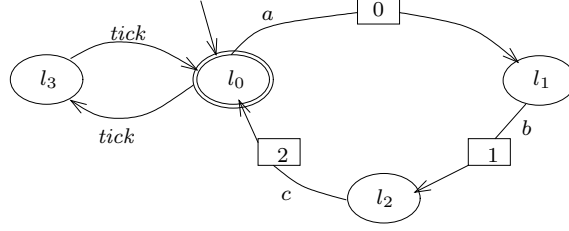


Fig. 3. Automaton \mathcal{A}_2 with delay blocks.

untimed language $\mathcal{U}(\mathcal{A}_1)$ is $\{a^n b^n c^n \mid n \geq 0\}$. Including the output time stamps in the words, we get the timed language $\mathcal{L}(\mathcal{A}_1)$ as $\{\langle a, 0 \rangle^n \langle b, 1 \rangle^n \langle c, 2 \rangle^n \mid n \geq 0\}$ (no output symbols appear after time 2). \square

Example 3. Consider the ADB \mathcal{A}_2 in Figure 3. The initial state is l_0 , which is also the only accepting state. The accepting runs of the automaton correspond to the regular expression sequence $(abc(tick\ tick)^*)^*$. Consider a particular run sequence $r = abcabc\ tick\ tick\ abc\ tick\ tick\ tick\ tick\ abc$. Recall that time advances in ADBs either via the explicit *tick* transition, or when the run ends in an accepting state. In the run r , the first two a occurrences are generated at time 0 as no *tick* transitions have been encountered until then; these two a occurrences appear immediately in the output at time 0 (the associated delay is 0 for the delay block). The first two b occurrences are also generated at time 0, but appear in the output at time 1, when the first *tick* transition is taken. The first two c occurrences are generated at time 0, and appear in the output at time 2, when the second *tick* transition is taken. Thus, after the first two *tick* transitions, the time-stamped output string is $\langle a, 0 \rangle^2 \langle b, 1 \rangle^2 \langle c, 2 \rangle^2$. The third a occurrence in r is generated at time 2 (after the first two *tick* transitions), and appears immediately at time 2. The third b occurrence in r is generated at time 2, and appears after a delay of one time unit, when the third *tick* transition is taken. The third c occurrence in r is generated at time 2, and appears at time 4, when the fourth *tick* transition is taken. Continuing in this fashion, we see that the time-stamped output corresponding to the run r is $\langle a, 0 \rangle^2 \langle b, 1 \rangle^2 \langle c, 2 \rangle^2 \langle a, 2 \rangle \langle b, 3 \rangle \langle c, 4 \rangle \langle a, 6 \rangle \langle b, 7 \rangle \langle c, 8 \rangle$.

Letting $\langle \sigma, i \rangle^0$ denote the empty string, the timed language of the automaton \mathcal{A}_2 can be observed to be

$$\mathcal{L}(\mathcal{A}_2) = \left\{ \begin{array}{l} \langle a, 0 \rangle^{n_0} \langle b, 1 \rangle^{n_0} \langle c, 2 \rangle^{n_0} \langle a, 2 \rangle^{n_2} \langle b, 3 \rangle^{n_2} \langle c, 4 \rangle^{n_2} \dots \\ \langle a, 2k \rangle^{n_{2k}} \langle b, 2k+1 \rangle^{n_{2k}} \langle c, 2k+2 \rangle^{n_{2k}} \\ \text{such that} \\ n_i \geq 0 \text{ for all } i, \text{ and } k \geq 0 \end{array} \right\}$$

The untimed language of the automaton \mathcal{A}_2 can be observed to be

$$\mathcal{U}(\mathcal{A}_2) = \left\{ \begin{array}{l} a^{n_0} b^{n_0} c^{n_0} a^{n_1} b^{n_1} c^{n_1} \dots \\ a^{n_k} b^{n_k} c^{n_k} \end{array} \middle| n_i \geq 0 \text{ for all } i \text{ and } k \geq 0 \right\}$$

Equivalently, using the untimed language of the automaton \mathcal{A}_1 from the previous example, $\mathcal{U}(\mathcal{A}_2) = \{w_0 w_1 \dots w_n \mid w_i \in \mathcal{U}(\mathcal{A}_1) \text{ for } 0 \leq i \leq n\}$. \square

Our contributions. In this work, along with the introduction of ADBs, we study their expressive power, closure properties, and the basic decision and model checking problems. Our main results are as follows:

- ★ *Expressive power:* We show that the untimed languages of ADBs strictly subsume regular languages, and are incomparable in expressive power to context-free languages. ADBs are able to express a simple class of languages not expressible by context-free languages. For example, the automata \mathcal{A}_1 of Figure 2 has the untimed language $\{a^n b^n c^n \mid n \geq 0\}$ which is not context free.
- ★ *Closure properties:* We show that untimed ADB languages are closed under union, concatenation, Kleene star, and intersection with regular languages, but not under complementation and intersection with other untimed ADB languages.
- ★ *Decision and model checking problems:* We show that the emptiness and the membership problems are decidable for ADBs in polynomial time, whereas the universality of untimed ADB languages is undecidable. Finally, we consider the model checking problem, where an ADB is considered as the model generating words, and a regular language specifies the desired set of words. The model checking problem is then the containment of the untimed ADB language in the regular language, and we show that the problem is PSPACE-complete.

Thus, ADBs provide a natural and practical extension of finite state automata for modeling discrete time processes involving delays where the output generation is via a regular process. ADBs though incomparable in expressiveness to context-free languages, enjoy several nice properties similar to that of context-free languages, for instance, ADBs admit decidable emptiness, membership and model checking algorithms. We note that the delays used in ADBs are of most use in *modelling* and *analysis* of naturally occurring delays in physical systems, not in directly *building* engineering systems. Thus, non-closure under intersection of ADBs is not a deal-breaker — systems are built compositionally as regular automata; delays are only used in the analysis of the composed system.

For our technical contribution we present illustrative ideas behind two of the key results. (1) We show that the balanced parenthesis language is not expressible as an untimed ADB language. This is a bit surprising because ADBs can express non context-free languages like $a^n b^n c^n$. This inexpressibility (which establishes incomparability to context-free languages) is a result of the fact that the maximum delay present in an ADB limits the “depth” of the nestings in the generated word. Consider a word $a^n \circ w \circ b^n$, where \circ is the concatenation operator, and w is a subword. To match the a^n with the b^n , the ADB needs to use at least one delay block, say of delay k . Then, to express matchings in the word w , it can only use delay blocks of delay *strictly less than* k . (2) We can model check an untimed ADB language against a regular specification (*i.e.* a finite state automaton). To show this, we check for emptiness of an untimed ADB language and a regular language complement of the specification by constructing a non-deterministic finite state automaton which has an accepting path iff the intersection of the languages is non-empty. This automaton maintains a guess of the future executions of the regular specification automaton for M future timepoints, where M is the largest delay of the given ADB. The guesses are verified whenever time advances. Proofs omitted from the main paper can be found in the appendix.

Related Work. The model of timed automata [AD94] is a widely studied formalism for timed systems. Timed automata do not have any construct for delaying generated output symbols, and their untimed languages are regular, unlike for ADBs. In the task scheduling context, a model which is somewhat related has recently been introduced in [Sti+11], the digraph real-time task model (DRT). In a DRT instance, jobs are released according to a specified directed weighted graph, where the weights on the edges denote the time that must elapse in between the job releases. The nodes, which correspond to jobs, are annotated with the worst case execution times and the deadlines for the jobs. Thus, the deadline sequence for when the jobs must finish differs from the jobs release sequence due to the deadline and execution time “delays”. However, the edge weights in

the DRT model are *strictly* positive and integer valued — this implies that the “queue” of currently executing jobs has length at most N where N can be computed from the DRT instance. Thus, the deadline sequences form a regular set. In ADBs, an *unbounded* number of symbols can be generated, before an output symbol is seen, thus the implicit queue is of unbounded length. This additional power of ADBs can be used to model scheduling problems where a bound on the number of job creations per unit time is not known a priori. The work in [MP95] explores delays in circuits. Only delays signals which “hold” for a given time d are of relevance, where d is a given constant; signals which do not persist for at least d time units are not output. This gives regularity, allowing the system to be modeled as a timed automaton. We do not require a hold time, in our discrete time framework, an unlimited number of letters (actually all) in between two time ticks are delayed if so specified.

The ADB model also has some similarity to computational models of automata augmented with queues. An ADB with M delay blocks can be viewed as writing to M unbounded queues at any given point in time, corresponding to the M delays indexed by the delay blocks. The work in [Iba00] presents decidability results for reader-writer systems augmented with one unbounded queue in between the reader and writer for communication, one pushdown stack for either the reader or writer, and finitely many reversal bounded counters for both. It also shows undecidability for two finite state automata (reader and writer) with two unbounded communication queues in between. The work of [BP83] shows decidability results for two finite state automata augmented with an unbounded one way communication queue in between them, and mention undecidability if there are more than two communicating finite state automata augmented with just one unbounded queue in the system. The work of [BH98] presents symbolic semi-algorithms for analyzing communicating finite state automata with queue communication channels. If the queue channels are *lossy*, then decidability can be shown for a variety of problems [AJ96]. Model checking is usually done on systems with *bounded* buffers (see *e.g.* [FM07]), and suffers from the state explosion problem with increasing buffer size. Our key result shows that the ADB model has the decidable model checking property in spite of containing any number of unbounded *delay* buffers. One key intuition behind the decidable result is the fact that messages corresponding to time Δ are invisible to an observer until all messages corresponding to the previous time-points have been output and consumed.

2 Automata with Delay Blocks

In this section we introduce our model of automata with delay blocks, and illustrate with examples the timed and untimed languages generated by these automata.

Automata with delay blocks (ADB). A finite *automata with delay blocks* (ADB) is a tuple $\mathcal{A} = (L, D, \Sigma, \delta, l_s, L_f)$ where

- L is a finite set of locations.
- $l_s \in L$ is the starting location.
- $L_f \subseteq L$ is the set of accepting locations.
- Σ is the set of output symbols.
- D is a finite set of delay blocks. Each delay block $d \in D$ is indexed by a natural number $t \geq 0$ to indicate the amount of delay for the outputs. We denote a delay block with delay t by \boxed{t} .
- δ is the transition relation,

$$\delta : \left((L \times \Sigma \times D) \cup (L \times \{\epsilon, tick\}) \right) \mapsto 2^L$$

where ϵ denotes the empty string, and $tick \notin \Sigma$ denotes a time passage of one time unit.

- A transition $\delta(l, \sigma, \boxed{t}) = L'$ denotes a location change from l to a location in L' non-deterministically, with σ being output t time units into the future.
- A transition $\delta(l, \epsilon) = L'$ denotes an epsilon transition from l to a location in L' non-deterministically, with no new output requirements.
- A transition $\delta(l, tick) = L'$ denotes time advancing by one time unit, and the location changing from l to a location in L' non-deterministically.

A finite string w is a sequence of elements. Given a string w , we let $|w|$ denote the length of the string w , and let $w[i]$ denote the i -th element (starting from index 0) in the string w if $|w| > i$. The empty string is denoted by ϵ . The concatenation of two words w_1 and w_2 is denoted $w_1 w_2$ and also $w_1 \circ w_2$. We also use the standard regular expression constructs. For $i \geq 0$, we denote by $\text{repeat}_i(w)$ the string w repeated i times (letting $\text{repeat}_0(w) = \epsilon$), i.e., $\text{repeat}_i(w)$ is the string $\underbrace{ww \dots w}_i$.

i occurrences

Discrete Timed words. A (discrete) timed word w is a finite string belonging to $(\Sigma \times \mathbb{N})^*$ where \mathbb{N} denotes the set of natural numbers. We refer to the first element of the tuple $w[i]$ as the *output symbol* and the second element of the tuple $w[i]$ as the *timestamp*. The timestamps denote the discrete time at which the first element of the tuples appear in the word. We require that for $i < j < |w|$, and for $w[i] = \langle w_i^\sigma, w_i^t \rangle$ and $w[j] = \langle w_j^\sigma, w_j^t \rangle$, we have $w_i^t \leq w_j^t$ (i.e. the timestamps are non-decreasing). Given a timed word $w \in (\Sigma \times \mathbb{N})^*$, let $\text{untime}(w) \in \Sigma^*$ be the untimed word denoting the projection of w onto Σ^* , that is, if $w = \langle \sigma_0, t_0 \rangle \dots \langle \sigma_m, t_m \rangle$, then $\text{untime}(w) = \sigma_0 \dots \sigma_m$. Given a timed word $w = \langle \sigma_0, t_0 \rangle \dots \langle \sigma_n, t_n \rangle$ and a natural number $\Delta \geq 0$, we let $w \oplus \Delta$ be the timed word $\langle \sigma_0, t_0 + \Delta \rangle \dots \langle \sigma_n, t_n + \Delta \rangle$ (the time stamps are advanced by Δ for all $w[i]$). Given an untimed word $w = \sigma_0 \sigma_1 \dots \sigma_m$, let $\kappa_t(w)$ denote the timed word $\langle \sigma_0, t \rangle \langle \sigma_1, t \rangle \dots \langle \sigma_m, t \rangle$, that is the timed word where each output symbol of w occurs at time t .

Generation of discrete timed words by ADBs. A generating run r of the automaton \mathcal{A} is a finite sequence $l_0 \xrightarrow{\alpha_0} l_1 \xrightarrow{\alpha_1} \dots l_n$ for $\alpha_i \in \{\epsilon, tick\} \cup (\Sigma \times D)$, such that l_0 is the starting location, l_n is an accepting location and $l_{i+1} \in \delta(l_i, \alpha_i)$ for $0 \leq i \leq n-1$. The automaton \mathcal{A} *outputs* or *generates* the timed word w if there exists a generating run $l_0 \xrightarrow{\alpha_0} l_1 \xrightarrow{\alpha_1} \dots l_n$ such that $\text{outword}(\alpha_0 \dots \alpha_n) = w$ where, informally, the $\text{outword}()$ function timestamps the output symbols according to their generation and delay block times, and arranges them in the proper timestamp order. A delay block \boxed{j} delays the output symbol by j time units. At time $t \in \mathbb{N}$ in a run, a delay block \boxed{j} can be considered to be feeding symbols to a queue \mathcal{Q}_{t+j} which will output the stored symbols at time $t+j$ (there is only one queue corresponding to an output time t). A *tick* transition explicitly advances time by one time unit. We also have that once the automaton stops at a final state, time automatically advances with symbols stored in the queues being output at the appropriate times. We note that time advances *only* at *tick* transitions, or when the automaton comes to rest at a final state.

Formally, $\text{outword}(\alpha_0 \dots \alpha_n)$ is the unique timed word w belonging to $(\Sigma \times \mathbb{N})^*$ defined as follows. For $\bar{\alpha} = \langle \alpha_0 \dots \alpha_n \rangle$, and $\langle \alpha_i^\sigma, \boxed{\alpha_i^t} \rangle \in \{\alpha_0, \dots, \alpha_n\}$, let $\text{wtime}(\langle \alpha_i^\sigma, \boxed{\alpha_i^t} \rangle, \bar{\alpha}) = \alpha_i^t + t_i$, where t_i denotes the number of occurrences of *tick* in $\alpha_0, \dots, \alpha_{i-1}$. Intuitively, the σ -element α_i^σ of each $\langle \alpha_i^\sigma, \boxed{\alpha_i^t} \rangle \in \{\alpha_0, \dots, \alpha_n\}$, appears exactly once in w , with $\text{wtime}(\langle \alpha_i^\sigma, \boxed{\alpha_i^t} \rangle, \bar{\alpha})$ denoting its timestamp. Formally, $\text{outword}(\alpha_0 \dots \alpha_n)$ is the unique timed word w such that

- $|w|$ is equal to the number of times symbols from $\Sigma \times \mathbb{N}$ appear in the string $\alpha_0 \alpha_1 \dots \alpha_n$.

- For all $i < |w|$, we have $w[i] = \langle \alpha_j^\sigma, \text{wtime}(\langle \alpha_j^\sigma, \boxed{\alpha_j^t} \rangle, \bar{\alpha}) \rangle$ where $\langle \alpha_j^\sigma, \boxed{\alpha_j^t} \rangle = \alpha[j]$ is such that for all k and $\alpha[k] = \langle \alpha_k^\sigma, \boxed{\alpha_k^t} \rangle$, the following conditions hold.

1. If either

- $k < j$ and $\text{wtime}(\alpha[k], \bar{\alpha}) \leq \text{wtime}(\alpha[j], \bar{\alpha})$; or
- $k > j$ and $\text{wtime}(\alpha[k], \bar{\alpha}) < \text{wtime}(\alpha[j], \bar{\alpha})$,

then for some $i' < i$, we have $w[i'] = \langle \alpha_k^\sigma, \text{wtime}(\alpha[k], \bar{\alpha}) \rangle$.

2. If either

- $k < j$ and $\text{wtime}(\alpha[k], \bar{\alpha}) > \text{wtime}(\alpha[j], \bar{\alpha})$; or
- $k > j$ and $\text{wtime}(\alpha[k], \bar{\alpha}) \geq \text{wtime}(\alpha[j], \bar{\alpha})$,

then for some $i' > i$, we have $w[i'] = \langle \alpha_k^\sigma, \text{wtime}(\alpha[k], \bar{\alpha}) \rangle$.

Thus, the placement of the σ -element of each $\alpha[j]$ is in increasing order of the timestamps $\text{wtime}(\alpha[j], \bar{\alpha})$, and if $\alpha[j]$ and $\alpha[k]$ result in the same timestamp, then the relative ordering is dictated by the relative ordering between j and k .

An equivalent alternative algorithmic definition of the function `outword()` is given in Function 1 with `StableSortTime` being a stable sorting function which sorts based on the second element of tuples¹.

Input : A string α from $((\Sigma \times D) \cup \{\epsilon, \text{tick}\})^*$

Output: A timed word w in $(\Sigma \times \mathbb{N})^*$

$w = \epsilon$;

$\text{curr_time} = i = j = 0$;

while $i < |\alpha|$ **do**

switch $\alpha[i]$ **do**

case ϵ

$i := i + 1$;

case tick

$i := i + 1$;

$\text{curr_time} := \text{curr_time} + 1$;

case $\langle \sigma, \boxed{m} \rangle$

$w[j] = \langle \sigma, \text{curr_time} + m \rangle$;

$i := i + 1$;

$j := j + 1$;

end

end

return `StableSortTime`(w);

Function `outword`(α)

Output languages of ADBs. The timed output *language* of \mathcal{A} is denoted by $\mathcal{L}(\mathcal{A})$ where $\mathcal{L}(\mathcal{A}) = \{w \mid w \text{ is a timed word generated by } \mathcal{A}\}$. For a timed language \mathcal{L} , we let $\text{untime}(\mathcal{L}) = \{\text{untime}(w) \mid w \in \mathcal{L}\}$. We also let $\mathcal{U}(\mathcal{A})$ denote the untimed language $\text{untime}(\mathcal{L}(\mathcal{A}))$. We have already illustrated

¹ Stable sorting algorithms maintain the original relative ordering of elements with equal key values.

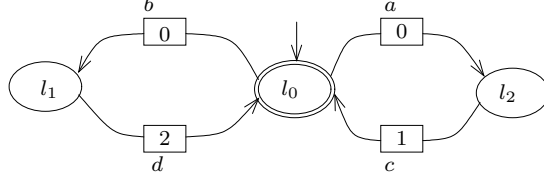


Fig. 4. Automaton \mathcal{A}_3 with delay blocks.

languages of ADBs with two examples in the introduction. Below we present another illustrating example.

Example 4. Consider the ADB \mathcal{A}_3 in Figure 4 with the initial state l_0 which is also the only final state. There are no explicit time advancing *tick* transitions in \mathcal{A}_3 , thus, time only advances once the generating run stops at an accepting state. Any generating run thus occurs at time 0, *i.e.* all the transitions are taken at time 0. Consider a generating run specified by the transition sequence $r = a c a c b d a c b d$. The symbols output at time 0 are the ones corresponding to delay blocks $\boxed{0}$, and these are (in order) $aabab$. The symbols output at time 1 are the ones corresponding to delay blocks $\boxed{1}$, and these are (in order) ccc . The symbols output at time 2 are the ones corresponding to delay blocks $\boxed{2}$, and these are (in order) dd . Thus, the output timestamped string corresponding to the transition sequence r is $\langle a, 0 \rangle^2 \langle b, 0 \rangle \langle a, 0 \rangle \langle b, 0 \rangle \langle c, 1 \rangle^3 \langle d, 2 \rangle^2$. It can be observed that the the timed language $\mathcal{L}(\mathcal{A}_3)$ is

$$\mathcal{L}(\mathcal{A}_3) = \left\{ w \mid \begin{array}{l} w \in (\langle a, 0 \rangle + \langle b, 0 \rangle)^* \langle c, 1 \rangle^* \langle d, 2 \rangle^* \text{ and} \\ |w|_{\langle a, 0 \rangle} = |w|_{\langle c, 1 \rangle} \text{ and} \\ |w|_{\langle b, 0 \rangle} = |w|_{\langle d, 2 \rangle} \end{array} \right\}$$

where $|w|_{\langle \sigma, i \rangle}$ denotes the number of occurrences of $\langle \sigma, i \rangle$ in w . The untimed language of \mathcal{A}_3 is obtained by projecting the timed words in the timed language $\mathcal{L}(\mathcal{A}_3)$ onto the output alphabet. We have

$$\mathcal{U}(\mathcal{A}_3) = \{u \mid u \in (a + b)^* c^* d^* \text{ and } |u|_a = |u|_c \text{ and } |u|_b = |u|_d\};$$

where $|u|_\sigma$ denotes the number of occurrences of σ in u . □

ADB Languages obtained from the outword operation on Regular Languages. Given an ADB $\mathcal{A} = (L, D, \Sigma, \delta, l_s, L_f)$ with the output symbol set Σ and M_d as the largest index for a delay block, there exists a corresponding regular finite automaton $\text{reg}(\mathcal{A}) = (L, \Sigma^\circ, \delta^\circ, l_s, L_f)$ over the delay-stamped symbol set $\Sigma^\circ = \Sigma \times \{0, \dots, M_d\} \cup \{\text{tick}, \epsilon\}$ such that

1. $\delta^\circ(l, \langle \sigma, t \rangle) = \delta(l, \sigma, \boxed{t})$
2. $\delta^\circ(l, \epsilon) = \delta(l, \epsilon)$.
3. $\delta^\circ(l, \text{tick}) = \delta(l, \text{tick})$

Intuitively, $\text{reg}(\mathcal{A})$ is just the ADB \mathcal{A} “interpreted” as a regular automaton. The regular language of $\text{reg}(\mathcal{A})$ is denoted by $\mathcal{R}(\mathcal{A})$. We define $\text{outword}(\mathcal{R}(\mathcal{A}))$ to be the timed word language $\{\text{outword}(w) \mid w \in \mathcal{R}(\mathcal{A})\}$.

Proposition 1. *Let \mathcal{A} be an ADB, and let $\text{reg}(\mathcal{A})$ be the corresponding regular finite automaton with the corresponding regular language $\mathcal{R}(\mathcal{A})$. We have $\mathcal{L}(\mathcal{A}) = \text{outword}(\mathcal{R}(\mathcal{A}))$.*

Proof. By definition. □

3 Expressiveness of Untimed Languages of ADBs

In this section we compare the expressive power untimed languages of ADBs against regular and context free languages. Given a regular or pushdown automaton \mathcal{A} without timed delay blocks, we let $\mathcal{U}(\mathcal{A})$ be the language of \mathcal{A} . First we show that ADBs can be considered to be a generalization of regular automata.

Proposition 2 (Generalization of regular automata). *Let \mathcal{A} be a regular automaton without timed delay blocks. Consider the ADB \mathcal{A}' obtained from \mathcal{A} such that (1) \mathcal{A}' has the same set of locations, set of accepting locations and starting location as \mathcal{A} ; and (2) the transition function $\delta^{\mathcal{A}'}$ is such that $\delta^{\mathcal{A}'}(l, \langle \sigma, 0 \rangle) = \delta^{\mathcal{A}}(l, \sigma)$ and $\delta^{\mathcal{A}'}(l, \epsilon) = \delta^{\mathcal{A}}(l, \epsilon)$. Then, $\mathcal{U}(\mathcal{A}) = \mathcal{U}(\mathcal{A}')$.*

Proof. The ADB \mathcal{A}' has no tick transitions. Thus since \mathcal{A}' only has delay blocks of duration 0, given a run r of \mathcal{A}' , the symbols from Σ are output in the order in which they are encountered in the run r . By construction, there is a one to one correspondence between the runs of \mathcal{A}' and \mathcal{A} such that the output symbol sequence in a run r of \mathcal{A}' is the same as the output symbol sequence in the corresponding run of \mathcal{A} . Hence, $\mathcal{U}(\mathcal{A}) = \mathcal{U}(\mathcal{A}')$. \square

We next show that the expressive power of untimed languages of ADBs is incomparable to that of context free languages.

Proposition 3. *Let \mathcal{U}^\dagger be the untimed language*

$$\mathcal{U}^\dagger = \left\{ \begin{array}{c} a^{n_1} \# a^{n_2} \# \dots \# a^{n_m} \# b^{n_m} \# b^{n_{m-1}} \# \dots \# b^{n_2} \# b^{n_1} \\ \text{such that} \\ m \geq 0 \text{ and } n_i \geq 1 \text{ for } 1 \leq i \leq m \end{array} \right\}.$$

There is no ADB \mathcal{A} such that $\mathcal{U}(\mathcal{A}) = \mathcal{U}^\dagger$

Proof. Intuitively, the proof below shows that the maximum delay present in an ADB limits the “depth” of the nestings in the generated word.

We prove by contradiction. Let $\mathcal{A} = (L, D, \Sigma, \delta, l_s, L_f)$ be any ADB with $\{a, b, \#\}$ as the set of output symbols such that $\mathcal{U}(\mathcal{A}) = \mathcal{U}^\dagger$. The automaton \mathcal{A} has a natural graph representation with the nodes in the graph corresponding to locations and the edges corresponding to δ . Let $G = \{S_1, \dots, S_p\}$ denote the set of strongly connected components (SCCs) of \mathcal{A} which are reachable from l_s , can reach a final location, and which contain at least one a -edge. Observe that every SCC S_i in G :

1. Must have a b -edge, and
2. Cannot have a *tick*-edge (for then an a can be made to appear after a b).
3. Every b -delay in the SCC must be greater than every a -delay (otherwise an a can be made to appear after a b since there are no *tick* edges).

Consider an SCC S from G . Let $p = l_0 l_1 \dots l_x$ be any path from l_s to an accepting location which passes through S . For $0 \leq i \leq j \leq x$, let $p[i]$ denote the state l_i and $p[i..j]$ the sub-path $l_i \dots l_j$. Let $p[\alpha..\beta]$ be a (maximal) sub-path which lies entirely within S , i.e. such that (1) $\alpha = 0$ or $p[\alpha - 1] \notin S$, and (2) $\beta = x$ or $p[\beta + 1] \notin S$. Let $C_\alpha^1, \dots, C_\alpha^q$ be all the non-overlapping cycles (each C_α^k has only one cycle) in $p[\alpha..\beta]$ such that the C_α^j occurs after C_α^i for $j > i$. The sub-path $p[\alpha..\beta]$ can then be written as

$$p[\alpha] p[\alpha + 1] \dots p[e_1] (C_\alpha^1)^{h_1} p[f_1] \dots p[e_2] (C_\alpha^2)^{h_2} \dots (C_\alpha^q)^{h_q} p[f_q] \dots p[\beta]$$

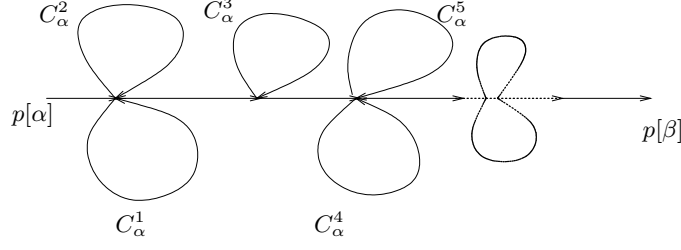


Fig. 5. Structure of sub-path $p[\alpha..\beta]$.

for some $e_1, f_1, h_1 \dots e_q, f_q, h_q$ such that the following subpaths of $p[\alpha..\beta]$ are all cycle free: (1) the path $p[\alpha]p[\alpha+1] \dots p[e_1]$; (2) the paths $p[f_{k-1}] \dots p[e_k]$ for all $2 \leq k \leq q$; and (3) the path $p[f_q] \dots p[\beta]$. The structure of the subpath $p[\alpha..\beta]$ is illustrated in Figure 5. Thus, C_α^1 is the first cycle in the $p[\alpha..\beta]$ sub-path, C_α^2 is the next cycle and so on. Observe that we may have $C_\alpha^i = C_\alpha^j$ for $i \neq j$.

Observe that each cycle C_α^i must have an a -transition (otherwise two consecutive $\#$ symbols can be made to appear in the output). Note that for any path $p[\alpha..\beta]$ and any such subcycle of the path, we must have that the number of a -edges in the subcycle equals the number of b -edges (otherwise we can “pump” the cycle to get a ’s that are unmatched by b ’s; as the matching must occur inside the the same cycle). Thus, each C_α^i can generate some $a^k..b^k$ pair. Consider any C_α^i . Let the maximum a -delay be Δ_a^i in C_α^i , and let the minimum b -delay be Δ_b^i . For any $j \neq i$, we must have one of the following to hold:

1. Either $\Delta_a^i = \Delta_a^j$ and $\Delta_b^i \neq \Delta_b^j$;
2. Or $\Delta_b^i = \Delta_b^j$ and $\Delta_a^i \neq \Delta_a^j$; or
3. $\Delta_b^i \neq \Delta_b^j$ and $\Delta_a^i \neq \Delta_a^j$.

For otherwise, if $\Delta_a^i = \Delta_a^j$ and $\Delta_b^i = \Delta_b^j$ the two cycles will generate an untimed subpart $a^{n_{k_1}}\#a^{n_{k_2}}\#b^{n_{k_1}}\#b^{n_{k_2}}$ (with enough pumping), when they should be generating the string with the b ’s switched (*i.e.* the string $a^{n_{i_1}}\#a^{n_{i_2}}\#b^{n_{i_2}}\#b^{n_{i_1}}$). Thus, in any accepting path $p[\alpha..\beta]$ via the SCC S , we can only have at most M^2 subcycles $C_\alpha^1, \dots, C_\alpha^q$, *i.e.* $q \leq M^2$ (as there are only at most M^2 distinct values of $\langle \Delta_a, \Delta_b \rangle$ tuples).

Since the maximum delay is finite (say $M-1$), the cycles in S can generate at most M^2 pairs of unbounded numbers of a ’s and b ’s. That is, the SCC S cannot generate the untimed language

$$\left\{ \begin{array}{c} a^{n_1}\#a^{n_2}\#\dots\#a^{n_{M^2+1}}\#b^{n_{M^2+1}}\#b^{n_{M^2}}\#\dots\#b^{n_2}\#b^{n_1} \\ \text{such that} \\ n_i \geq 1 \text{ for } 1 \leq i \leq M^2 + 1 \end{array} \right\}.$$

Hence, if there are K SCCs in \mathcal{A} , then \mathcal{A} cannot generate the untimed language

$$\left\{ \begin{array}{c} a^{n_1}\#a^{n_2}\#\dots\#a^{n_{M^2+K+1}}\#b^{n_{M^2+K+1}}\#b^{n_{M^2+K}}\#\dots\#b^{n_2}\#b^{n_1} \\ \text{such that} \\ n_i \geq 1 \text{ for } 1 \leq i \leq M^2 + K + 1 \end{array} \right\}.$$

Hence, it follows that there does not exist an ADB \mathcal{A} such that $\mathcal{U}(\mathcal{A}) = \mathcal{U}^\dagger$. \square

Proposition 4 (Incomparability with Pushdown Automata). *The following assertions hold.*

1. *There exists an ADB \mathcal{A} such that $\mathcal{U}(\mathcal{A})$ is not context free.*

2. There exists a visibly pushdown automata \mathcal{A} such that there is no ADB \mathcal{A}' with $\mathcal{U}(\mathcal{A}) = \mathcal{U}(\mathcal{A}')$.

Proof. For the first part of the proposition, consider the ADB \mathcal{A}_1 of Figure 2. The untimed language of \mathcal{A}_1 is $\{a^n b^n c^n \mid n \geq 0\}$ which is not context free.

The second part of the theorem follows from Proposition 3, noting that there exists a visibly pushdown automaton which generates the language \mathcal{U}^\dagger . \square

Proposition 4 shows that there is a tradeoff between the expressive power of ADBs and pushdown automata. On one hand, ADBs are not restricted to matching only once (i.e., they can generate $a^n b^n c^n$), but on the other they lose the infinite nesting capability of pushdown automata (e.g., in the language \mathcal{U}^\dagger of Proposition 3).

Theorem 1 (Expressive power of ADBs). *The following assertions hold: (1) The class of untimed languages of ADBs strictly subsumes the class of regular languages. (2) The class of untimed languages of ADBs is incomparable in expressive power as compared to the class of context-free languages.*

Proof. The results follow from Propositions 2 and 4. \square

4 Closure Properties

In this section we will study the closure properties of timed and untimed languages of ADBs with respect to operations like union, intersection, complement, concatenation and Kleene star.

Proposition 5 (Closure under union). *Let \mathcal{A}_1 and \mathcal{A}_2 be ADBs. There exists an ADB \mathcal{A} such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2)$.*

Proof. The ADB \mathcal{A} has a special initial states, and two ϵ transitions from this initial state to copies of \mathcal{A}_1 and \mathcal{A}_2 . \square

Proposition 6 (Closure under intersection with regular languages). *Given an untimed ADB language \mathcal{U} and a regular language \mathcal{R} , the language $\mathcal{U} \cap \mathcal{R}$ is an untimed ADB language.*

Proof. Given an ADB \mathcal{A}_1 with \mathcal{U} and a finite-state automata \mathcal{A}_2 for a regular language \mathcal{R} , we will present an explicit construction of an ADB with untimed language $\mathcal{U} \cap \mathcal{R}$ in Proposition 15. The desired result will follow from the construction. \square

Concatenation and Kleene star. We will now consider closure under concatenation and Kleene star. Given untimed languages \mathcal{U} , \mathcal{U}_1 and \mathcal{U}_2 , we define their concatenation $\mathcal{U}_1 \circ \mathcal{U}_2$ and Kleene star \mathcal{U}^* as follows:

$$\begin{aligned}\mathcal{U}_1 \circ \mathcal{U}_2 &\triangleq \{w_1 \circ w_2 \mid w_1 \in \mathcal{U}_1 \text{ and } w_2 \in \mathcal{U}_2\} \\ \mathcal{U}^* &\triangleq \{\epsilon\} \cup \{w_1 \circ w_2 \circ \dots \circ w_i \mid i \in \mathbb{N}, 1 \leq j \leq i, w_j \in \mathcal{U}\}\end{aligned}$$

Proposition 7 (Closure under concatenation). *Let \mathcal{U}_1 and \mathcal{U}_2 be untimed ADB languages. Then $\mathcal{U}_1 \circ \mathcal{U}_2$ is an untimed ADB language.*

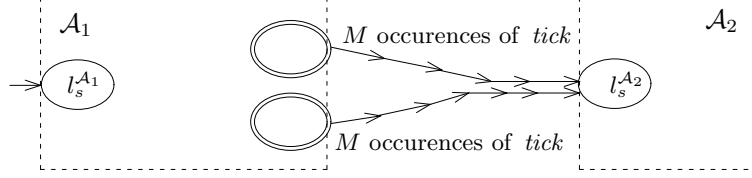


Fig. 6. Automaton \mathcal{A}_1 and \mathcal{A}_2 concatenated to get $\mathcal{U}_1 \circ \mathcal{U}_2$.

Proof. Let $\mathcal{U}_1 = \mathcal{U}(\mathcal{A}_1)$ and $\mathcal{U}_2 = \mathcal{U}(\mathcal{A}_2)$. Let the largest delay in the delay blocks of \mathcal{A}_1 be M . Consider the ADB \mathcal{A} in Figure 6. The ADB \mathcal{A} has M occurrences of *tick* transitions from every final location of \mathcal{A}_1 to the starting location of \mathcal{A}_2 . Let r_1 and r_2 be accepting runs in \mathcal{A}_1 and \mathcal{A}_2 which result in the untimed output words w_1 and w_2 respectively. Let $r_1 = l_0 \xrightarrow{\alpha_0} l_1 \xrightarrow{\alpha_1} \dots l_n$. Then, $r_1^\dagger = l_0 \xrightarrow{\alpha_0} l_1 \xrightarrow{\alpha_1} \dots l_n \xrightarrow{\text{tick}} l_{n_1} \xrightarrow{\text{tick}} l_{n_2} \dots l_{n_M}$ forms a part of an accepting run in \mathcal{A} , namely the

run $r_1^\dagger \circ \xrightarrow{\epsilon} \circ r_2$. Moreover, the run r_1^\dagger generates the untimed output word $w_1 \circ w_2$, as the maximum delay of a delay block in r_1 is M , thus all the output symbols have been output before the r_2 part in r_1^\dagger starts.

In the other direction, given an accepting run r_1^\dagger of \mathcal{A} , it can be decomposed into $r_1 \circ \underbrace{\xrightarrow{\text{tick}} l_{n_1} \xrightarrow{\text{tick}} l_{n_2} \dots l_{n_M}}_{M \text{ occurrences of tick}} \circ r_2$ such that r_1 and r_2 are accepting runs in \mathcal{A}_1 and \mathcal{A}_2 respectively.

The result follows from noting that r_1^\dagger generates the untimed output word $w_1 \circ w_2$ where w_1 and w_2 are untimed output words generated by the runs r_1 and r_2 respectively. \square

Proposition 8 (Closure under Kleene star). *Let \mathcal{U} be an untimed ADB language. Then \mathcal{U}^* is an untimed ADB language.*

Proof. Let $\mathcal{U} = \mathcal{U}(\mathcal{A})$. Consider the ADB \mathcal{A}_* in Figure 7. $l_s^{\mathcal{A}}$ is the starting location of \mathcal{A} , and $l_s^{\mathcal{A}_*}$

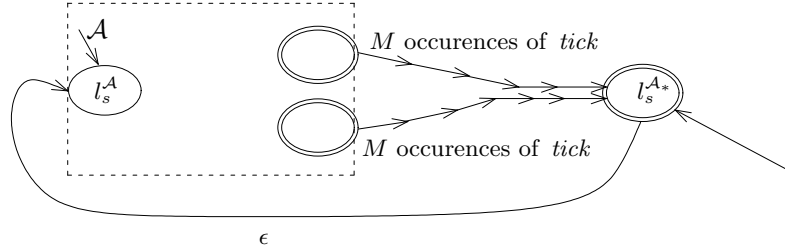


Fig. 7. Automaton \mathcal{A}_* using \mathcal{A} to get \mathcal{U}^* .

is the starting location of \mathcal{A}_* . The ADB \mathcal{A}_* has M occurrences of *tick* transitions from every final location of \mathcal{A} to the starting location of \mathcal{A}_* . There is also an ϵ transition from the starting location of \mathcal{A}_* to the starting location of \mathcal{A} . It can be seen that the untimed output language of \mathcal{A}_* is \mathcal{U}^* . The proof follows along similar lines to the proof of Proposition 7. \square

We will now show the ADB languages are not closed under some Boolean operations, and towards this goal we first prove a pumping lemma.

Proposition 9 (Pumping Lemma for ADB runs). *Let \mathcal{A} be an ADB and let L be the set of locations of \mathcal{A} . Let $w \in \mathcal{L}(\mathcal{A})$ with $|w| > |L|$ be the output timed word corresponding to an accepting run $r = l_0 \xrightarrow{\alpha_0} l_1 \xrightarrow{\alpha_1} \dots l_n$. Consider any subrun r_s of r , i.e. $r = r_0 \circ r_s \circ r_1$, such that r_s contains at least $|L|$ transitions. Then, there exists a subrun r_p of r_s , i.e. $r_s = r_{s_0} \circ r_p \circ r_{s_1}$ with r_p containing at most $|L|$ transitions such that for all $i \geq 0$ the runs $r_0 \circ r_{s_0} \circ \text{repeat}_i(r_p) \circ r_{s_1} \circ r_1$ are also accepting runs of \mathcal{A} .*

Proof. The proof follows from the pumping lemma for regular finite state automata, and from Proposition 1. \square

Remark 1. There are difficulties in obtaining a pumping lemma for timed words. We give an example. Let $r, r_0, r_{s_0}, r_p, r_{s_1}, r_1$ be as in Proposition 9. Let w be the timed word corresponding to the run r . Let $\langle \sigma_\alpha, t_\alpha \rangle$ and $\langle \sigma_\beta, t_\beta \rangle$ be the timestamped symbols generated by some transition in $r_0 \circ r_{s_0}$, and by some transition in $r_{s_1} \circ r_1$ respectively. Let us denote these two transitions as tr_α and tr_β . We may have $t_\alpha > t_\beta$, i.e. $\langle \sigma_\alpha, t_\alpha \rangle$ appears after $\langle \sigma_\beta, t_\beta \rangle$ in the timed word w , even though the transition which generates $\langle \sigma_\alpha, t_\alpha \rangle$ occurs before the transition which generates $\langle \sigma_\beta, t_\beta \rangle$. Let the number of *tick* transitions in r_p be Δ_p . Each “pump” of r_p introduces an additional delay of Δ_p between when the transitions tr_α and tr_β occur. Eventually, after enough pumps, the delay will be large enough that the timestamped output symbol corresponding to tr_β will appear after the timestamped output symbol corresponding to tr_α . Thus, when we pump an accepting run, the resulting timed word, with each pump, may undergo a *reordering* of the output symbols corresponding to the unpumped run parts. There is also a reordering corresponding to the pumped run part. \square

Proposition 10 (Non-closure under intersection). *There exist ADBs \mathcal{A}_1 and \mathcal{A}_2 such that (1) $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$ is not an ADB language, and (2) $\mathcal{U}(\mathcal{A}_1) \cap \mathcal{U}(\mathcal{A}_2)$ is not an untimed ADB language.*

Proof. (Sketch.) Consider the language

$$\mathcal{L}^\dagger = \left\{ \kappa_0(w\#)\kappa_1(w\#)\kappa_2(w\#)\dots\kappa_n(w\#) \left| \begin{array}{l} w \in \{a, b\}^* \text{ and} \\ n \geq 0 \end{array} \right. \right\}$$

where $\kappa_i()$ is the function defined in Section 2. We show \mathcal{L}^\dagger is not an ADB language, and that there exist ADBs \mathcal{A}_1 and \mathcal{A}_2 such that $\mathcal{L}^\dagger = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$. To show the first claim, let \mathcal{L}^\dagger be the output language of an ADB \mathcal{A}^\dagger containing K locations. Consider a timed word $w_\dagger = \kappa_0(w\#)\kappa_1(w\#)\kappa_2(w\#)\dots\kappa_{K+2}(w\#)$ with $|w| > K$. Let r_\dagger be the generating run for w_\dagger . Using the pumping lemma, we can show there exists a subrun r_p of r_\dagger such that (1) the subrun r_p contains at least one output symbol transition, and (2) the subrun contains at most K output symbol transitions; and (3) for $r_\dagger = r_0 \circ r_p \circ r_1$, we have that $r_0 \circ r_1$ is also a generating run for \mathcal{A}^\dagger (i.e., we pump down r_p). Let w_{01} be the output word corresponding to the generating run $r_0 \circ r_1$. Because of the constraints on r_p , we have that w_{01} contains at least one, and at most K output symbols less than w . It can be checked that this means that w_{01} is not a member of \mathcal{L}^\dagger , a contradiction. To show that \mathcal{L}^\dagger is the intersection of two ADB languages, we consider two ADBs, the first ADB checks that the word with timestamp $2j$ matches the word with time $2j + 1$ for all j ; the second ADB checks that the word with timestamp $2j + 1$ matches the word with time $2j + 2$ for all j . It can be checked that such ADBs exist and that the intersection of the languages is \mathcal{L}^\dagger . \square

Proposition 11 (Non-closure under complementation). *Given an ADB \mathcal{A} , let $\overline{\mathcal{L}}(\mathcal{A})$ denote the complement language of $\mathcal{L}(\mathcal{A})$. There exists an ADB \mathcal{A} such that for all ADBs \mathcal{A}' we have $\overline{\mathcal{L}}(\mathcal{A}) \neq \mathcal{L}(\mathcal{A}')$, and $\overline{\mathcal{U}}(\mathcal{A}) \neq \mathcal{U}(\mathcal{A}')$. \square*

We summarize our results in the following theorem.

Theorem 2 (Closure properties). *The class of untimed languages of ADBs are closed under union, concatenation, Kleene star, and intersection with regular languages, but not closed under intersection and complementation.* \square

5 Decision Problems and Model Checking

In this section we first study the decision problems such as emptiness, universality for ADBs, and then study the model checking problem. In the model checking problem we consider an ADB as the model to generate words, and a specification given as regular language. Our goal is to check the containment of the untimed language of the ADB in the regular language.

5.1 Decision Problems

Proposition 12 (Emptiness checking). *Given an ADB \mathcal{A} , it can be checked in linear time whether $\mathcal{L}(\mathcal{A}) = \emptyset$.*

Proof. The proposition follows from the fact that $\mathcal{L}(\mathcal{A})$ is non-empty iff there is a path from the initial location to an accepting location. \square

Proposition 13 (Membership checking of timed words). *Given an ADB \mathcal{A} with $n_{\mathcal{A}}$ locations and $m_{\mathcal{A}}$ edges, and a timed word w , checking whether $w \in \mathcal{L}(\mathcal{A})$ can be checked in time $O(M \cdot (n_{\mathcal{A}} + m_{\mathcal{A}} + |w| + T_e))$, where T_e is the largest timestamp in w , and M is the largest delay of a delay block in \mathcal{A} (thus, if M is a constant and $T_e = O(n_{\mathcal{A}} + m_{\mathcal{A}} + |w|)$, then we have a linear time algorithm).*

Proof. (Sketch.) Let $w = \langle w_0^\sigma, w_0^t \rangle \langle w_1^\sigma, w_1^t \rangle \dots \langle w_n^\sigma, w_n^t \rangle$. Let the end timestamp of w be T_e (i.e. $w_n^t = T_e$). We first construct a finite state deterministic regular automaton \mathcal{A}_w with just one path (corresponding to w) over the alphabet $\Sigma_w = \{tick, \langle w_0^\sigma, w_0^t \rangle, \langle w_1^\sigma, w_1^t \rangle \dots \langle w_n^\sigma, w_n^t \rangle\}$. We then construct a (non-deterministic) ADB \mathcal{A}^\dagger based on \mathcal{A} and \mathcal{A}_w such that \mathcal{A}^\dagger has an accepting path iff \mathcal{A} outputs the timed word w . Let M be the largest delay of a delay block in \mathcal{A} . The automaton \mathcal{A}^\dagger will simulate the executions of \mathcal{A} ; and of \mathcal{A}_w simultaneously for the current time, and for time upto M time units in the future. That is, the automaton \mathcal{A}^\dagger is able to verify that \mathcal{A}_w first generates the output symbols corresponding to the current time outputs in \mathcal{A} , then generate output symbols corresponding to current time plus one in \mathcal{A} , and so on. The details of the construction are omitted for lack of space. \square

Proposition 14 (Universality). *Let \mathcal{A} be an ADB. It is undecidable to check whether $\mathcal{U}(\mathcal{A}) = \Sigma^*$.*

Proof. Let \mathcal{T} be a Turing machine with Σ as the tape alphabet. Let a valid computation of \mathcal{T} be denoted by an untimed string $w = w_0 \# w_1 \# \dots w_n$ such that w_0 represents the initial tape configuration, w_{i+1} is a tape configuration that follows from w_i for $i \geq 0$, and $\#$ is a special delimiter symbol. We first show that there exists an ADB \mathcal{A} such that $\mathcal{U}(\mathcal{A})$ is the set of strings denoting the invalid computations of \mathcal{T} .

If a string w represents an invalid computation, then one of the following conditions must hold.

1. The string w is not of the form $w_0 \# w_1 \# \dots w_n$, where each w_i denotes a tape configuration.

2. w_0 is not an initial tape configuration.
3. w_n is not an accepting tape configuration.
4. w_{i+1} does not follow from w_i for some i .

The set of strings satisfying conditions 1,2 or 3 is regular. There exists an ADB \mathcal{A}' such that $\mathcal{U}(\mathcal{A}')$ is the set of strings satisfying the last condition. The automaton \mathcal{A}' first generates strings from $(\Sigma \cup \{\#\})^*$ at time 0. It then non-deterministically moves to a location from which it generates $w_i\#w_{i+1}$ such that (a) w_i is a configuration (*i.e.*, a string from $\Sigma^*Q\Sigma^*$ where Q is the set of locations of \mathcal{T}), (b) the configuration w_i is generated at time 0 and w_{i+1} generated at time 1, and (c) w_{i+1} is not a configuration that follows from w_i (this can be done by “knowing” some future two symbols of w_i at time 0, and accordingly generating a symbol at time 1 for w_{i+1} such that w_{i+1} cannot be a configuration following w_i). Once such $w_i\#w_{i+1}$ is generated, \mathcal{A}' then generates strings from $(\Sigma \cup \{\#\})^*$ at time 2. Since ADBs are closed under union, we can take the union of ADBs generating untimed strings satisfying either of the four conditions. The ADBs for conditions 1,2 and 3 “operate” at times 3, 4 and 5 respectively (*i.e.* they are regular automata with delay blocks of 3,4 and 5 respectively at every transition). This union ADB \mathcal{A} will then generate all untimed strings denoting invalid computations of \mathcal{T} .

Now, if were decidable to check whether $\mathcal{U}(\mathcal{A}) = \Sigma^*$, then it would mean we can check whether the language of \mathcal{T} is non-empty, as the language of \mathcal{T} is non-empty iff there exists a valid computation of \mathcal{T} , and a valid computation of \mathcal{T} exists iff $\mathcal{U}(\mathcal{A}) \neq \Sigma^*$. Thus, if \mathcal{A} is an ADB, it is undecidable in general to check whether $\mathcal{U}(\mathcal{A}) = \Sigma^*$. \square

Corollary 1 (Equivalence to a regular language). *Let \mathcal{A} be an ADB and let \mathcal{R} be a regular language. It is undecidable to check whether $\mathcal{U}(\mathcal{A}) = \mathcal{R}$.* \square

Corollary 2 (Containment in another delay model). *Let \mathcal{A} and \mathcal{A}' be ADBs. It is undecidable to check whether $\mathcal{U}(\mathcal{A}) \subseteq \mathcal{U}(\mathcal{A}')$.*

Proof. We reduce universality of untimed languages of ADBs to this problem. Let \mathcal{A} be an ADB such that $\mathcal{U}(\mathcal{A}) = \Sigma^*$. Then, given any ADB \mathcal{A}' , we have $\mathcal{U}(\mathcal{A}) \subseteq \mathcal{U}(\mathcal{A}')$ iff $\mathcal{U}(\mathcal{A}') = \Sigma^*$. \square

Theorem 3 (Decision problems). *The following assertions hold for timed and untimed ADB languages:*

1. *The emptiness checking can be achieved in linear time, and the membership checking can also be achieved in linear time if the largest delay of the delay blocks is constant, and the largest timestamp in the word is of the order of the length of the word plus the automaton size.*
2. *The universality, containment in other untimed ADB languages, and equivalence to regular languages are undecidable.*

Proof. The first item follows from Proposition 12 and Proposition 13. The second item follows from Proposition 14, Corollaries 1 and 2. \square

5.2 Model Checking

In this section we study the containment of a given untimed language of an ADB within a given regular language.

Proposition 15 (Checking emptiness of intersection with a regular language). *Let \mathcal{A} be an ADB, and let \mathcal{A}_r be a regular finite state automaton. It is decidable to check emptiness of*

$\mathcal{U}(\mathcal{A}) \cap \mathcal{U}(\mathcal{A}_r)$ in time $O(n_{\mathcal{A}} \cdot n_{\mathcal{A}_r}^{2M+1} + m_{\mathcal{A}} \cdot m_{\mathcal{A}_r}^{M+1} + m_{\mathcal{A}} \cdot n_{\mathcal{A}_r})$ where $n_{\mathcal{A}}, n_{\mathcal{A}_r}$ are the numbers of locations in $\mathcal{A}, \mathcal{A}_r$ and $m_{\mathcal{A}}, m_{\mathcal{A}_r}$ the numbers of edges in $\mathcal{A}, \mathcal{A}_r$ respectively, and M is the largest delay of a delay block in \mathcal{A} .

Proof. We construct a non-deterministic ADB \mathcal{A}^\dagger such that $\mathcal{U}(\mathcal{A}^\dagger) = \mathcal{U}(\mathcal{A}) \cap \mathcal{U}(\mathcal{A}_r)$ and then apply Proposition 12.

Let M be the largest delay of a delay block in \mathcal{A} . The automaton \mathcal{A}^\dagger will simulate the executions of \mathcal{A} ; and of \mathcal{A}_r simultaneously for the current time, and for time upto M time units in the future. That is, the automaton \mathcal{A}^\dagger is able to verify that some execution of \mathcal{A}_r is such that \mathcal{A}_r first generates the output symbols corresponding to the current time outputs in \mathcal{A} , then generates symbols corresponding to current time plus one in \mathcal{A} , and so on. To concurrently simulate executions of \mathcal{A}_r corresponding to $M + 1$ time points, the automaton will maintain a tuple of locations. The tuple will have $2M + 2$ components:

1. The first component will correspond to a location of \mathcal{A} , and is used to simulate executions of \mathcal{A} .
2. The next $M + 1$ components will correspond to locations of \mathcal{A}_r used for concurrently simulating \mathcal{A}_r corresponding to the current time point, and the next M time points.
3. The final M components correspond to the “guesses” on the locations of \mathcal{A}_r for the final locations of the initial and following $M - 1$ timepoints. Whenever time elapses in \mathcal{A} via an explicit *tick* transition, we verify that our “guesses” for the ending locations of \mathcal{A}_r are correct.

Formally, let $\mathcal{A} = (L^{\mathcal{A}}, D^{\mathcal{A}}, \Sigma, \delta^{\mathcal{A}}, l_s^{\mathcal{A}}, L_f^{\mathcal{A}})$ and $\mathcal{A}_r = (L^{\mathcal{A}_r}, \Sigma, \delta^{\mathcal{A}_r}, l_s^{\mathcal{A}_r}, L_f^{\mathcal{A}_r})$. The ADB \mathcal{A}^\dagger is as follows:

- The location set $L^{\mathcal{A}^\dagger}$ is $L^{\mathcal{A}} \times (L^{\mathcal{A}_r})^{2M+1} \cup \{l_s^{\mathcal{A}^\dagger}\}$ where M is the largest delay of a delay block in \mathcal{A} , and $l_s^{\mathcal{A}^\dagger}$ is a new location.
- The initial location is $l_s^{\mathcal{A}^\dagger}$.
- The transition function $\delta^{\mathcal{A}^\dagger}$ is as follows:

$$\bullet \delta^{\mathcal{A}^\dagger}(l_s^{\mathcal{A}^\dagger}, \epsilon) = \left\{ \langle l_s^{\mathcal{A}}, l_s^{\mathcal{A}_r}, l_1^{\mathcal{A}_r}, \dots, l_M^{\mathcal{A}_r}, l_1^{\mathcal{A}_r}, \dots, l_M^{\mathcal{A}_r} \rangle \mid \begin{array}{l} l_s^{\mathcal{A}} \text{ is the initial location of } \mathcal{A}, \\ l_s^{\mathcal{A}_r} \text{ is the initial location of } \mathcal{A}_r \\ \text{and } l_j^{\mathcal{A}_r} \in L \text{ for } M \geq j \geq 1 \end{array} \right\}$$

The locations $l_s^{\mathcal{A}_r}, l_1^{\mathcal{A}_r}, \dots, l_M^{\mathcal{A}_r}$ for \mathcal{A}_r correspond the starting location for the current time, and for the following M time points. The guessed locations $l_1^{\mathcal{A}_r}, \dots, l_M^{\mathcal{A}_r}$ are explicitly stored as the last M components (to be verified later). There is an ϵ -transition from $l_s^{\mathcal{A}^\dagger}$ to various locations corresponding to all the possible guesses for the starting locations for times 1 through M .

$$\bullet \delta^{\mathcal{A}^\dagger}(\langle l_0^{\mathcal{A}}, l_0^{\mathcal{A}_r}, l_1^{\mathcal{A}_r}, \dots, l_M^{\mathcal{A}_r}, \tilde{l}_1^{\mathcal{A}_r}, \dots, \tilde{l}_M^{\mathcal{A}_r} \rangle, \sigma, \boxed{t}) =$$

$$\left\{ \langle l_0'^{\mathcal{A}}, l_0'^{\mathcal{A}_r}, l_1'^{\mathcal{A}_r}, \dots, l_M'^{\mathcal{A}_r}, \tilde{l}_1^{\mathcal{A}_r}, \dots, \tilde{l}_M^{\mathcal{A}_r} \rangle \mid \begin{array}{l} l_0'^{\mathcal{A}} \in \delta^{\mathcal{A}}(l_0^{\mathcal{A}}, \langle \sigma, \boxed{t} \rangle), \\ l_t^{\mathcal{A}_r} \in \delta^{\mathcal{A}_r}(l_t^{\mathcal{A}_r}, \sigma), \text{ and} \\ l_j^{\mathcal{A}_r} = l_j^{\mathcal{A}_r} \text{ for } j \neq t \end{array} \right\}$$

This transition corresponds to the case when \mathcal{A} transitions on $\langle \sigma, \boxed{t} \rangle$ from location $l_0^{\mathcal{A}}$. In the location $\langle l_0^{\mathcal{A}}, l_0^{\mathcal{A}_r}, l_1^{\mathcal{A}_r}, \dots, l_M^{\mathcal{A}_r}, \tilde{l}_1^{\mathcal{A}_r}, \dots, \tilde{l}_M^{\mathcal{A}_r} \rangle$ of \mathcal{A}^\dagger , the component corresponding to t time units in the future is updated. The location component of \mathcal{A} is also updated. The rest of the components remain the same.

- $\delta^{\mathcal{A}^\dagger}(\langle l_0^{\mathcal{A}}, l_0^{\mathcal{A}_r}, l_1^{\mathcal{A}_r}, \dots, l_M^{\mathcal{A}_r}, \tilde{l}_1^{\mathcal{A}_r}, \dots, \tilde{l}_M^{\mathcal{A}_r} \rangle, \epsilon) =$

$$\left\{ \langle l_0'^{\mathcal{A}}, l_0'^{\mathcal{A}_r}, l_1'^{\mathcal{A}_r}, \dots, l_M'^{\mathcal{A}_r}, \tilde{l}_1^{\mathcal{A}_r}, \dots, \tilde{l}_M^{\mathcal{A}_r} \rangle \mid \begin{array}{l} l_0'^{\mathcal{A}} \in \delta^{\mathcal{A}}(l_0^{\mathcal{A}}, \epsilon) \cup \{l_0^{\mathcal{A}}\} \text{ and} \\ l_j'^{\mathcal{A}_r} \in \delta^{\mathcal{A}_r}(l_j^{\mathcal{A}_r}, \epsilon) \cup \{l_j^{\mathcal{A}_r}\} \text{ for } M \geq j \geq 0 \end{array} \right\}$$

The ϵ -transitions of \mathcal{A}^\dagger correspond to the case when either \mathcal{A} or \mathcal{A}_r make ϵ -transitions. The automaton \mathcal{A}_r can make ϵ -transitions either at the current time, or be supposed to make them in the future.

- $\delta^{\mathcal{A}^\dagger}(\langle l_0^{\mathcal{A}}, l_0^{\mathcal{A}_r}, l_1^{\mathcal{A}_r}, \dots, l_M^{\mathcal{A}_r}, \tilde{l}_1^{\mathcal{A}_r}, \dots, \tilde{l}_M^{\mathcal{A}_r} \rangle, tick) =$

* If $l_0^{\mathcal{A}_r} = \tilde{l}_1^{\mathcal{A}_r}$ then

$$\left\{ \langle l_0'^{\mathcal{A}}, l_0'^{\mathcal{A}_r}, l_1'^{\mathcal{A}_r}, \dots, l_M'^{\mathcal{A}_r}, \tilde{l}_1^{\mathcal{A}_r}, \dots, \tilde{l}_M^{\mathcal{A}_r} \rangle \mid \begin{array}{l} l_0'^{\mathcal{A}} \in \delta^{\mathcal{A}}(l_0^{\mathcal{A}}, tick), \\ l_j'^{\mathcal{A}_r} = l_{j+1}^{\mathcal{A}_r} \text{ and } \tilde{l}_j^{\mathcal{A}_r} = \tilde{l}_{j+1}^{\mathcal{A}_r} \text{ for } 0 \leq j \leq M-1, \\ l_M'^{\mathcal{A}_r} \in L^{\mathcal{A}_r} \text{ and } \tilde{l}_M^{\mathcal{A}_r} \in L^{\mathcal{A}_r} \text{ with } l_M'^{\mathcal{A}_r} = \tilde{l}_M^{\mathcal{A}_r} \end{array} \right\}$$

* \emptyset otherwise.

When time advances from time t to $t+1$ in \mathcal{A}^\dagger (and in \mathcal{A}), we need to verify that the initial guess $\tilde{l}_1^{\mathcal{A}_r}$ was correct (recall that $\tilde{l}_1^{\mathcal{A}_r}$ was guessed previously in time t to be the location in \mathcal{A}_r at the beginning of time $t+1$). Also, since one time unit has passed, the guess $\tilde{l}_{j+1}^{\mathcal{A}_r}$ for the earlier $j+1$ -th future time unit now becomes the guess corresponding to j -th future time unit. Similarly, the location component $l_{j+1}^{\mathcal{A}_r}$ corresponding to the $j+1$ -th future time unit now becomes the location component corresponding to j -th future time unit. A new guess for the starting location for the M -th future time unit is also chosen.

– The set of final locations is

$$\left\{ \langle l_0^{\mathcal{A}}, l_0^{\mathcal{A}_r}, l_1^{\mathcal{A}_r}, \dots, l_M^{\mathcal{A}_r}, \tilde{l}_1^{\mathcal{A}_r}, \dots, \tilde{l}_M^{\mathcal{A}_r} \rangle \mid \begin{array}{l} l_0^{\mathcal{A}} \in L_f^{\mathcal{A}} \text{ and } l_M^{\mathcal{A}_r} \in L_f^{\mathcal{A}_r} \text{ and} \\ l_j^{\mathcal{A}_r} = \tilde{l}_{j+1}^{\mathcal{A}_r} \text{ for } 0 \leq j \leq M-1 \end{array} \right\}$$

\mathcal{A}^\dagger ensures that the automaton \mathcal{A}_r ends up in an accepting location at the end. Also, \mathcal{A}^\dagger checks that the guesses for the starting locations at each of the M future timepoints were correct.

The automaton \mathcal{A}^\dagger has an accepting run π^\dagger only if both of the conditions hold.

1. The run π^\dagger corresponds to a matching generating run π in \mathcal{A} .
2. The untimed word $\text{untime}(w_\pi)$ can be generated from \mathcal{A}_r , where w_π denotes the timed word output by \mathcal{A} corresponding to the generating path π . That is, the automaton \mathcal{A}_r generates the output symbols in w_π in the *right order* corresponding to $\text{untime}(w_\pi)$.

Thus, we have that $\mathcal{U}(\mathcal{A}^\dagger) = \mathcal{U}(\mathcal{A}) \cap \mathcal{U}(\mathcal{A}_r)$, and that $\mathcal{U}(\mathcal{A}) \cap \mathcal{U}(\mathcal{A}_r)$ is non-empty iff $\mathcal{U}(\mathcal{A}^\dagger)$ is non-empty. The number of locations in \mathcal{A}^\dagger is $1 + n_{\mathcal{A}} \cdot n_{\mathcal{A}_r}^{2M+1}$. The number of edges is $n_{\mathcal{A}_r}^M + m_{\mathcal{A}} \cdot m_{\mathcal{A}_r}^{M+1} + m_{\mathcal{A}} \cdot n_{\mathcal{A}_r}$. Thus, emptiness of $\mathcal{U}(\mathcal{A}) \cap \mathcal{U}(\mathcal{A}_r)$ can be checked in time $O(n_{\mathcal{A}} \cdot n_{\mathcal{A}_r}^{2M+1} + m_{\mathcal{A}} \cdot m_{\mathcal{A}_r}^{M+1} + m_{\mathcal{A}} \cdot n_{\mathcal{A}_r})$. \square

Theorem 4 (Model Checking). *Let \mathcal{A} be an ADB and \mathcal{A}_r be a regular non-deterministic finite-state automaton, and let $n_{\mathcal{A}}, n_{\mathcal{A}_r}$ be the numbers of locations in $\mathcal{A}, \mathcal{A}_r$ and $m_{\mathcal{A}}, m_{\mathcal{A}_r}$ be the numbers of edges in $\mathcal{A}, \mathcal{A}_r$ respectively, and M be the largest delay of a delay block in \mathcal{A} . The following assertions hold:*

1. The problem whether $\mathcal{U}(\mathcal{A}) \subseteq \mathcal{U}(\mathcal{A}_r)$ can be checked in time $O(n_{\mathcal{A}} \cdot 2^{n_{\mathcal{A}_r} \cdot (2M+1)} + m_{\mathcal{A}} \cdot 2^{m_{\mathcal{A}_r} \cdot (M+1)} + m_{\mathcal{A}} \cdot 2^{n_{\mathcal{A}_r}})$.
2. If M is constant, then the problem of whether $\mathcal{U}(\mathcal{A}) \subseteq \mathcal{U}(\mathcal{A}_r)$ is PSPACE-complete.

Proof. The result is obtained as follows: (i) we first obtain an automata for the complement of the language of \mathcal{A}_r (and the automata can be obtained by first determinizing \mathcal{A}_r and then complementing, and thus has at most $2^{n_{\mathcal{A}_r}}$ locations and $2^{m_{\mathcal{A}_r}}$ edges); and (ii) then check emptiness of intersection using Proposition 15. This gives the result for the first item. For the second item we note that if M is constant, then we obtain an exponential size automata and the emptiness check is achieved by checking reachability to a final location (in non-deterministic log-space over an exponential graph). This gives the PSPACE upper bound, and the PSPACE lower bound follows from the fact the containment is PSPACE hard for regular non-deterministic finite-state automata (which are special cases of ADBs). The desired result follows. \square

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6 Appendix

Proof of Proposition 10.

Proof. Consider the language

$$\mathcal{L}^\dagger = \{\kappa_0(w\#)\kappa_1(w\#)\kappa_2(w\#)\dots\kappa_n(w\#) \mid w \in \{a, b\}^* \text{ and } n \geq 0\}$$

We show \mathcal{L}^\dagger is not an ADB language, and that there exist ADBs \mathcal{A}_1 and \mathcal{A}_2 such that $\mathcal{L}^\dagger = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$.

1. (\mathcal{L}^\dagger is not an ADB language.)

Suppose the claim is false, *ie.* let $\mathcal{L}^\dagger = \mathcal{L}(\mathcal{A})$. Let the number of locations in \mathcal{A} be K . Consider a timed word $w_\dagger = \kappa_0(w\#)\kappa_1(w\#)\kappa_2(w\#)\dots\kappa_{K+2}(w\#)$ with $|w| > K$. Let r_\dagger be the generating run for w_\dagger . Using the pumping lemma for ADB runs (Proposition 9), we can show there exists a subrun r_p of r_\dagger such that (1) the subrun r_p contains at least one output symbol transition, and (2) the subrun contains at most K output symbol transitions; and (3) for $r_\dagger = r_0 \circ r_p \circ r_1$, we have that $r_0 \circ r_1$ is also a generating run for \mathcal{A}^\dagger (*i.e.*, we pump down r_p). Let w_{01} be the output word corresponding to the generating run $r_0 \circ r_1$. Because of the constraints on r_p , we have that w_{01} contains at least one, and at most K output symbols less than w . It can be checked that this means that w_{01} is not a member of \mathcal{L}^\dagger , a contradiction. Thus, \mathcal{L}^\dagger is not an ADB language.

2. (\mathcal{L}^\dagger is the intersection of two ADB languages).

Consider the following two languages \mathcal{L}_1^\dagger and \mathcal{L}_2^\dagger :

$$\mathcal{L}_1^\dagger = \left\{ \begin{array}{l} \kappa_0(\overline{w}_0\#)\kappa_1(\overline{w}_0\#)\kappa_2(\overline{w}_2\#)\kappa_3(\overline{w}_2\#)\dots \\ \kappa_{2n}(\overline{w}_{2n}\#)\kappa_{2n+1}(\overline{w}_{2n}\#) \end{array} \middle| \overline{w}_j \in \{a, b\}^* \text{ for all } j \text{ and } n \geq 0 \right\} \cup \left\{ \begin{array}{l} \kappa_0(\overline{w}_0\#)\kappa_1(\overline{w}_0\#)\kappa_2(\overline{w}_2\#)\kappa_3(\overline{w}_2\#)\dots \\ \kappa_{2n-2}(\overline{w}_{2n-2}\#)\kappa_{2n-1}(\overline{w}_{2n-1}\#)\kappa_{2n}(\overline{w}_{2n}\#) \end{array} \middle| \overline{w}_j \in \{a, b\}^* \text{ for all } j \text{ and } n \geq 0 \right\} \quad (1)$$

and

$$\mathcal{L}_2^\dagger = \left\{ \begin{array}{l} \kappa_0(\overline{w}_{-1}\#)\kappa_1(\overline{w}_1\#)\kappa_2(\overline{w}_1\#) \\ \kappa_3(\overline{w}_3\#)\kappa_4(\overline{w}_3\#)\dots\kappa_{2n-1}(\overline{w}_{2n-1}\#)\kappa_{2n}(\overline{w}_{2n-1}\#) \end{array} \middle| \overline{w}_j \in \{a, b\}^* \text{ for all } j \text{ and } n \geq 0 \right\} \cup \left\{ \begin{array}{l} \kappa_0(\overline{w}_{-1}\#)\kappa_1(\overline{w}_1\#)\kappa_2(\overline{w}_1\#) \\ \kappa_3(\overline{w}_3\#)\kappa_4(\overline{w}_3\#)\dots \\ \kappa_{2n-1}(\overline{w}_{2n-1}\#)\kappa_{2n}(\overline{w}_{2n-1}\#)\kappa_{2n}(\overline{w}_{2n}\#) \end{array} \middle| \overline{w}_j \in \{a, b\}^* \text{ for all } j \text{ and } n \geq 0 \right\} \quad (2)$$

The language \mathcal{L}_1^\dagger contains all (discrete) times words of the form $\kappa_0(\overline{w}_0\#)\kappa_1(\overline{w}_1\#)\kappa_2(\overline{w}_2\#)\dots$ such that $\overline{w}_{2j} = \overline{w}_{2j+1}$ for all j . The language \mathcal{L}_2^\dagger contains all (discrete) times words of the form $\kappa_0(\overline{w}_0\#)\kappa_1(\overline{w}_1\#)\kappa_2(\overline{w}_2\#)\dots$ such that $\overline{w}_{2j+1} = \overline{w}_{2j+2}$ for all j .

We show both \mathcal{L}_1^\dagger and \mathcal{L}_2^\dagger to be ADB languages. Consider the ADB \mathcal{A}_1^\dagger in Figure 8 which has l_0 as the initial location and l_5 and l_7 as the accepting locations. It can be seen that $\mathcal{L}(\mathcal{A}_1^\dagger) = \mathcal{L}_1^\dagger$.

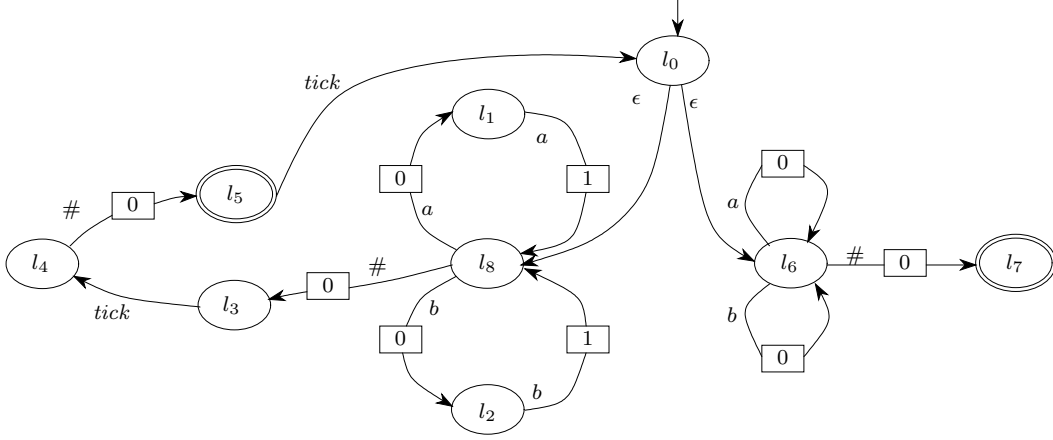


Fig. 8. Automaton \mathcal{A}_1^\dagger with delay blocks such that $\mathcal{L}(\mathcal{A}_1^\dagger) = \mathcal{L}_1^\dagger$.

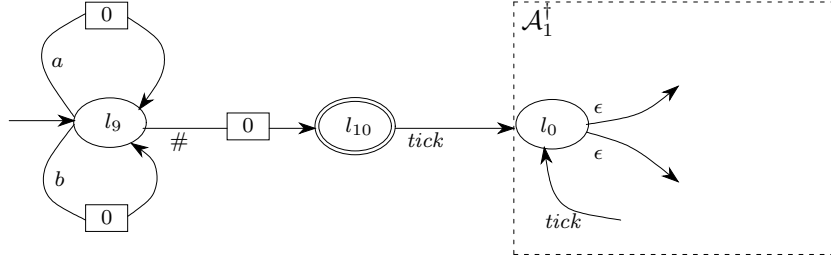


Fig. 9. Automaton \mathcal{A}_2^\dagger with delay blocks such that $\mathcal{L}(\mathcal{A}_2^\dagger) = \mathcal{L}_2^\dagger$.

Consider the right hand side of the automaton from the starting location l_0 . This portion of the ADB results in the output part $\{\kappa_0(\overline{w}) \mid \overline{w} \in \{a, b\}^*\}$ *i.e.* it is used to add the last “unmatched” segment in the second part after the union in Equation 1. The left hand side of the automaton is used to generate the segments $\kappa_{2i}(\overline{w}_{2i}\#)\kappa_{2i+1}(\overline{w}_{2i}\#)$.

The ADB \mathcal{A}_2^\dagger in Figure 9 has l_9 as the (only) starting location, and includes the locations and transitions of \mathcal{A}_1^\dagger as shown in the Figure. The *tick* transition from l_{10} goes to the location l_0 . The accepting locations are l_5 , l_7 and l_{10} . The automaton just “shifts” the checks of \mathcal{A}_1^\dagger by one time unit, by the transitions between l_9 and l_{10} . It can be seen that $\mathcal{L}(\mathcal{A}_2^\dagger) = \mathcal{L}_2^\dagger$.

The proof for untimed languages is similar to that for timed languages (we use the untimed language $\text{untime}(\mathcal{L}^\dagger)$). \square

Proof of Proposition 11.

Proof. Since $\mathcal{L}(\mathcal{A}_1 \cap \mathcal{A}_2) = \overline{\mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2)}$, and since ADBs are not closed under intersection (Proposition 10) but are closed under union (Proposition 5) we have that ADBs are not closed under complementation (a similar proof applies to untimed languages). \square

Proof of Proposition 13.

Proof. Let $w = \langle w_0^\sigma, w_0^t \rangle \langle w_1^\sigma, w_1^t \rangle \dots \langle w_n^\sigma, w_n^t \rangle$. Let the end timestamp of w be T_e (*i.e.* $w_n^t = T_e$). We first construct a finite state deterministic automaton \mathcal{A}_w with just one path (corresponding to w) over the alphabet $\Sigma_w = \{\text{tick}, \langle w_0^\sigma, w_0^t \rangle, \langle w_1^\sigma, w_1^t \rangle \dots \langle w_n^\sigma, w_n^t \rangle\}$ as follows.

- The set of locations is $\{l_0, l_1, \dots, l_{T_e}\} \cup \{l_{\langle w_0^\sigma, w_0^t \rangle}, l_{\langle w_1^\sigma, w_1^t \rangle} \dots l_{\langle w_n^\sigma, w_n^t \rangle}\}$.
- Let $w = \overline{w}_0 \circ \overline{w}_1 \circ \dots \circ \overline{w}_{T_e}$ where \overline{w}_k is the substring of w containing all the timestamped tuples of time k (let \overline{w}_k be ϵ if w does not contain any k -timestamped tuples). Let $i_k = \sum_{j=0}^k |\overline{w}_j|$ for $k = 0..T_e$. Note that if w contains a timestamp k , then $\langle w_{i_k}^\sigma, w_{i_k}^t \rangle$ is the last tuple timestamped with k . Thus,

$$w = \underbrace{\langle w_0^\sigma, w_0^t \rangle \langle w_1^\sigma, w_1^t \rangle \dots \langle w_{i_0}^\sigma, w_{i_0}^t \rangle}_{\text{timestamp 0}} \underbrace{\langle w_{i_0+1}^\sigma, w_{i_0+1}^t \rangle \dots \langle w_{i_1}^\sigma, w_{i_1}^t \rangle}_{\text{timestamp 1}} \dots \underbrace{\langle w_{i_{T_e-1}+1}^\sigma, w_{i_{T_e-1}+1}^t \rangle \dots \langle w_{i_{T_e}}^\sigma, w_{i_{T_e}}^t \rangle}_{\text{timestamp } T_e}$$

Note that $w_j^t = k$ for all $i_{k-1}+1 \leq j \leq i_k$ in the above representation. The finite state automaton \mathcal{A}_w will generate the language $\overline{w}_0 \circ \text{tick} \circ \overline{w}_1 \circ \text{tick} \circ \dots \text{tick} \circ \overline{w}_{T_e} \circ \text{tick}^*$.

- The starting location is l_0 and the only accepting location is $l_{\langle w_{i_{T_e}}^\sigma, w_{i_{T_e}}^t \rangle}$. We refer to the accepting location $l_{\langle w_{i_{T_e}}^\sigma, w_{i_{T_e}}^t \rangle}$ as the location l_\perp .
- The set of transitions is as follows. For all $k \geq -1$
 - If there is a timestamp $k+1$ in w , then there are the following transitions.
 - * The transition $l_{k+1} \xrightarrow{\langle w_{i_k+1}^\sigma, w_{i_k+1}^t \rangle} l_{\langle w_{i_k+1}^\sigma, w_{i_k+1}^t \rangle}$ (letting $i_{-1} = -1$).
 - * For all j such that $i_k + 1 < j \leq i_{k+1}$, we have the transition $l_{j-1} \xrightarrow{\langle w_j^\sigma, w_j^t \rangle} l_j$.
 - * If $k+1$ is not the greatest timestamp in w , then the transition $l_{\langle w_{i_k+1}^\sigma, w_{i_k+1}^t \rangle} \xrightarrow{\text{tick}} l_{k+2}$.
 - * The looping transition $l_{\langle w_{i_{T_e}}^\sigma, w_{i_{T_e}}^t \rangle} \xrightarrow{\text{tick}} l_{\langle w_{i_{T_e}}^\sigma, w_{i_{T_e}}^t \rangle}$.
 - If there is no timestamp $k+1$ in w , but there is a timestamp larger than $k+1$ in w , then the transition $l_{k+1} \xrightarrow{\text{tick}} l_{k+2}$.

It can be confirmed that the finite state automaton \mathcal{A}_w generates the strings $\overline{w}_0 \circ \text{tick} \circ \overline{w}_1 \circ \text{tick} \circ \dots \text{tick} \circ \overline{w}_{T_e} \circ \text{tick}^*$.

We construct a (non-deterministic) ADB \mathcal{A}^\dagger based on \mathcal{A} and \mathcal{A}_w such that \mathcal{A}^\dagger has an accepting path iff \mathcal{A} outputs the timed word w . Let M be the largest delay of a delay block in \mathcal{A} . The automaton \mathcal{A}^\dagger will simulate the executions of \mathcal{A} ; and of \mathcal{A}_w simultaneously for the current time, and for time upto M time units in the future. That is, the automaton \mathcal{A}^\dagger is able to verify that \mathcal{A}_w first generates the output symbols corresponding to the current time outputs in \mathcal{A} , then generates symbols corresponding to current time plus one in \mathcal{A} , and so on. To concurrently simulate executions of \mathcal{A}_w corresponding to $M+1$ time points, the automaton will maintain a tuple of locations. The tuple will have $M+2$ components:

1. The first component will correspond to a location of \mathcal{A} , and is used to simulate executions of \mathcal{A} .
2. The next $M+1$ components will correspond to locations of \mathcal{A}_w used for concurrently simulating \mathcal{A}_w corresponding to the current time point, and the next M time points.

Formally, let $\mathcal{A} = (L^{\mathcal{A}}, D^{\mathcal{A}}, \Sigma, \delta^{\mathcal{A}}, l_s^{\mathcal{A}}, l_f^{\mathcal{A}})$. The ADB \mathcal{A}^\dagger is as follows:

- The location set $L^{\mathcal{A}^\dagger}$ is $(L^{\mathcal{A}} \cup \{l_{\text{tick}}^{\mathcal{A}}\}) \times (L^{\mathcal{A}_w})^{M+1}$ where M is the largest delay of a delay block in \mathcal{A} , where $l_{\text{tick}}^{\mathcal{A}}$ is a new location component (*i.e.* it is not present in \mathcal{A}).
- The initial location is $\langle l_s^{\mathcal{A}}, l_0^{\mathcal{A}_w}, l_1^{\mathcal{A}_w}, \dots, l_M^{\mathcal{A}_w} \rangle$ (we have added the superscript \mathcal{A}_w to the locations of \mathcal{A}_w for clarity).

- The set of accepting locations is $\left(L_f^{\mathcal{A}} \cup \{l_{tick}^{\mathcal{A}}\}\right) \times \underbrace{\{l_{\perp}^{\mathcal{A}_w}\} \times \{l_{\perp}^{\mathcal{A}_w}\} \times \dots \{l_{\perp}^{\mathcal{A}_w}\}}_{M+1 \text{ occurrences}}.$

(Recall that $l_{\perp}^{\mathcal{A}_w}$ denotes the location $l_{\langle w_{T_e}^{\sigma}, w_{T_e}^t \rangle}$, and is the end sink accepting location of \mathcal{A}_w .)

- The transition function $\delta^{\ddagger}()$ is as follows.

- $\delta^{\mathcal{A}^{\ddagger}}(\langle l^{\mathcal{A}}, l_{\langle 0 \rangle}^{\mathcal{A}_w}, l_{\langle 1 \rangle}^{\mathcal{A}_w}, \dots, l_{\langle M \rangle}^{\mathcal{A}_w} \rangle, \epsilon) = \delta^{\mathcal{A}}(l^{\mathcal{A}}, \epsilon) \times \{l_{\langle 0 \rangle}^{\mathcal{A}_w}\} \times \{l_{\langle 1 \rangle}^{\mathcal{A}_w}\} \times \dots \{l_{\langle M \rangle}^{\mathcal{A}_w}\}$ (the locations $l_{\langle j \rangle}^{\mathcal{A}_w}$ are unrelated to the locations $l_j^{\mathcal{A}_w}$).
- $\delta^{\mathcal{A}^{\ddagger}}(\langle l^{\mathcal{A}}, l_{\langle 0 \rangle}^{\mathcal{A}_w}, l_{\langle 1 \rangle}^{\mathcal{A}_w}, \dots, l_{\langle M \rangle}^{\mathcal{A}_w} \rangle, \sigma, \boxed{t}) =$

$$\begin{cases} \delta^{\mathcal{A}}(l^{\mathcal{A}}, \sigma, \boxed{t}) \times \{l_{\langle 0 \rangle}^{\mathcal{A}_w}, l_{\langle 1 \rangle}^{\mathcal{A}_w}, \dots, l_{\langle t-1 \rangle}^{\mathcal{A}_w}\} \times \delta^{\mathcal{A}_w}(l_{\langle t \rangle}^{\mathcal{A}_w}, \sigma) \times \{l_{\langle t+1 \rangle}^{\mathcal{A}_w}, \dots, l_{\langle M \rangle}^{\mathcal{A}_w}\} & \text{if } l_{\langle 0 \rangle}^{\mathcal{A}_w} = l_{\langle x, k \rangle}^{\mathcal{A}_w}, \\ & \text{or } l_{\langle 0 \rangle}^{\mathcal{A}_w} = l_k^{\mathcal{A}_w}; \\ & \text{both with } k + t \leq T_e. \\ \emptyset & \text{otherwise.} \end{cases}$$

This corresponds to the fact that whenever \mathcal{A} has a delay transition σ, \boxed{t} , the automaton \mathcal{A}^{\ddagger} must ensure that the t -time component of \mathcal{A}_w takes a transition on the symbol σ , so long as \mathcal{A}_w has not seen the whole word w .

- $\delta^{\mathcal{A}^{\ddagger}}(\langle l^{\mathcal{A}}, l_{\langle 0 \rangle}^{\mathcal{A}_w}, l_{\langle 1 \rangle}^{\mathcal{A}_w}, \dots, l_{\langle M \rangle}^{\mathcal{A}_w} \rangle, tick)$ is as follows.

* Let $f(l^{\mathcal{A}})$ defined as

$$f(l^{\mathcal{A}}) = \begin{cases} \delta^{\mathcal{A}}(l^{\mathcal{A}}, tick) & \text{if } l^{\mathcal{A}} \text{ is not an accepting location of } \mathcal{A} \\ \delta^{\mathcal{A}}(l^{\mathcal{A}}, tick) \cup \{l_{tick}^{\mathcal{A}}\} & \text{if } l^{\mathcal{A}} \text{ is an accepting location of } \mathcal{A} \\ l_{tick}^{\mathcal{A}} & \text{if } l^{\mathcal{A}} = l_{tick}^{\mathcal{A}} \end{cases}$$

This corresponds to the fact that the accepting language of \mathcal{A} does not change if we add a *tick* transition from accepting locations to a new sink accepting location $l_{tick}^{\mathcal{A}}$. Doing this simplifies the sequel.

* Let $g(l_{\langle M \rangle}^{\mathcal{A}_w})$ be defined as

$$g(l_{\langle M \rangle}^{\mathcal{A}_w}) = \begin{cases} l_{k+1}^{\mathcal{A}_w} & \text{if } l_{\langle M \rangle}^{\mathcal{A}_w} = l_{\langle x, k \rangle}^{\mathcal{A}_w}, \text{ or } l_{\langle M \rangle}^{\mathcal{A}_w} = l_k^{\mathcal{A}_w}; \text{ both with } k < T_e \\ l_{\perp}^{\mathcal{A}_w} & \text{otherwise} \end{cases}$$

Recall that $l_{\perp}^{\mathcal{A}_w}$ denotes the location $l_{\langle w_{T_e}^{\sigma}, w_{T_e}^t \rangle}$, and is the end sink accepting location of \mathcal{A}_w . Thus, $g(l_{\langle M \rangle}^{\mathcal{A}_w})$ denotes the starting location $l_{k+1}^{\mathcal{A}_w}$ for time $k + 1$ if $l_{\langle M \rangle}^{\mathcal{A}_w} = l_{\langle x, k \rangle}^{\mathcal{A}_w}$, or $l_{\langle M \rangle}^{\mathcal{A}_w} = l_k^{\mathcal{A}_w}$; both with $k < T_e$. If $l_{\langle M \rangle}^{\mathcal{A}_w}$ corresponds to the T_e locations, then the $g(l_{\langle M \rangle}^{\mathcal{A}_w})$ is the sink accepting location $l_{\perp}^{\mathcal{A}_w}$.

* $\delta_{\mathcal{A}^\dagger}(\langle l^{\mathcal{A}}, l^{\mathcal{A}_w}_{\langle 0 \rangle}, l^{\mathcal{A}_w}_{\langle 1 \rangle}, \dots, l^{\mathcal{A}_w}_{\langle M \rangle} \rangle, tick)$ is then

$$\left\{ \begin{array}{l} \emptyset \\ f(l^{\mathcal{A}}) \times \{l^{\mathcal{A}_w}_{\langle 1 \rangle}\} \times \{l^{\mathcal{A}_w}_{\langle 2 \rangle}\} \times \dots \{l^{\mathcal{A}_w}_{\langle M \rangle}\} \times \{g(l^{\mathcal{A}_w}_{\langle M \rangle})\} \end{array} \right. \left(\begin{array}{l} \text{if } l^{\mathcal{A}_w}_{\langle 0 \rangle} \neq l^{\mathcal{A}_w}_{\langle w_{i_k}^\sigma, w_{i_k}^t \rangle} \text{ for every} \\ \quad 0 \leq k \leq T_e \\ \text{i.e., } l^{\mathcal{A}_w}_{\langle 0 \rangle} \text{ is not a location in } \mathcal{A}_w \\ \text{from which there exists an} \\ \text{outgoing } tick \text{ transition.} \\ \text{otherwise, i.e. } l^{\mathcal{A}_w}_{\langle 0 \rangle} \text{ is a location} \\ \text{in } \mathcal{A}_w \text{ from which there exists} \\ \text{an outgoing } tick \text{ transition.} \end{array} \right)$$

In this transition, the automaton \mathcal{A}^\dagger checks that for the time segment Δ corresponding to $\langle x, \Delta \rangle = \langle 0 \rangle$ or $\Delta = \langle 0 \rangle$, the automaton \mathcal{A}_w has finished reading all the symbols for that time segment. If so, the elements in the location tuple of \mathcal{A}^\dagger are left shifted, and the next starting location component corresponding to time $k + 1$ added. If $k + 1 > T_e$, then we just add the looping location $l^{\mathcal{A}_w}_\perp$.

Observe that if $l^{\mathcal{A}}$ is an accepting location, and $l^{\mathcal{A}_w}_{\langle 0 \rangle}, l^{\mathcal{A}_w}_{\langle 1 \rangle}, \dots, l^{\mathcal{A}_w}_{\langle M \rangle}$ all correspond to the locations denoting the ends of the corresponding time segments of w , and $l^{\mathcal{A}_w}_{\langle M \rangle} = l^{\mathcal{A}_w}_\perp$, then the *tick* transitions guarantee that the location $\langle l^{\mathcal{A}}, l^{\mathcal{A}_w}_\perp, l^{\mathcal{A}_w}_\perp, \dots, l^{\mathcal{A}_w}_\perp \rangle$ will be reached.

It can be checked that the ADB \mathcal{A}^\dagger is such that

$$\mathcal{L}(\mathcal{A}^\dagger) = \begin{cases} w & \text{if } w \in \mathcal{L}(\mathcal{A}) \\ \emptyset & \text{otherwise} \end{cases}$$

Thus, $w \in \mathcal{L}(\mathcal{A})$ iff $\mathcal{L}(\mathcal{A}^\dagger) \neq \emptyset$. To check for emptiness of $\mathcal{L}(\mathcal{A}^\dagger)$, we just have to do a reachability search from the initial location to the accepting locations. The number of locations of \mathcal{A}_w is $|w| + T_e$. The number of locations of \mathcal{A}^\dagger is $n_{\mathcal{A}} \cdot (|w| + T_e)^{M+1}$. The number of edges of \mathcal{A}^\dagger is $m_{\mathcal{A}} + T_e + |w|$. To check for reachability of the accepting locations, we do need to explicitly construct the automaton \mathcal{A}^\dagger , if we construct only the reachable from the initial state part using an on the fly algorithm, we will encounter at most $T_e + |w| + n_{\mathcal{A}}$ locations and $m_{\mathcal{A}} + T_e + |w|$ edges. To traverse an edge, we have to do $O(M)$ work, since a state of \mathcal{A}^\dagger is a $M + 2$ -tuple. Thus, the timed word membership test can be done in time $O(M \cdot (n_{\mathcal{A}} + m_{\mathcal{A}} + |w| + T_e))$. \square

Proposition 16. *Given three arbitrary ADBs $\mathcal{A}_1, \mathcal{A}_2$ and \mathcal{A}_3 , it is undecidable to check whether $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2) \cap \mathcal{L}(\mathcal{A}_3) = \emptyset$.*

Proof. The proof is similar to the proof of the corresponding result for context free grammars. Let \mathcal{T} be a Turing machine with Σ as the tape alphabet, Q as the set of locations, and F as the set of accepting locations. Let a valid computation of \mathcal{T} be denoted by an untimed string $w = w_0 \# w_1 \# \dots w_n$ such that w_0 represents the initial tape configuration, w_{i+1} is a tape configuration that follows from w_i for $i \geq 0$, and $\#$ is a special delimiter symbol. A valid computation of \mathcal{T} can be denoted by the timed word $\hat{w} = \kappa_0(w_0 \#) \kappa_1(w_1 \#) \kappa_2(w_2 \#) \dots \kappa_n(w_n \#)$. That is, \hat{w} denotes the timed word corresponding to w such that the substring $w_i \#$ occurs at time i . Observe that a timed word $\hat{w} = \kappa_0(w_0 \#) \kappa_1(w_1 \#) \kappa_2(w_2 \#) \dots \kappa_n(w_n \#)$ represents a valid computation of \mathcal{T} iff all the following conditions hold.

1. Each w_i represents a valid configuration of \mathcal{T} , that is $\kappa_i(w_i\#)$ belongs to the set $(\Sigma \times \{i\})^*(Q \times \{i\})(\Sigma \times \{i\})^*\#$.
2. w_0 represents the initial configuration of \mathcal{T} , that is $\kappa_i(w_0\#)$ is of the form $(\{q_0\} \times \{i\})(\Sigma \times \{i\})^*\#$ where q_0 is the initial location of \mathcal{T} .
3. w_n represents an accepting configuration of \mathcal{T} , that is $\kappa_n(w_n)$ is a string in $(\Sigma \times \{n\})^*(F \times \{n\})(\Sigma \times \{i\})^*$.
4. Each w_{i+1} represents a configuration of \mathcal{T} following from w_i .

We can construct an ADB \mathcal{A}_1 (having delay blocks only of delay 0) which generates strings satisfying conditions 1,2 and 3. We now show that we can construct ADBs \mathcal{A}_2 and \mathcal{A}_3 such that the strings in $\mathcal{L}(\mathcal{A}_2) \cap \mathcal{L}(\mathcal{A}_3)$ satisfy condition 4. The automaton \mathcal{A}_2 checks that the configuration w_{2i+1} occurring at time $2i+1$ follows from the configuration w_{2i} occurring at time $2i$ for $i \geq 0$. It does this by first generating two symbols from $w_{2i}\#$ at time $2i$, and then generating the appropriate first symbol for w_{2i+1} using a delay block of delay 1. It then repeatedly generates a symbol from $w_{2i}\#$, and the corresponding next symbol for w_{2i+1} using a delay block of delay 1. When the $\#$ symbol is generated for $w_{2i}\#$, there can be at most two symbols remaining for w_{2i+1} which are then generated. This procedure can be repeated for each i by a loop in \mathcal{A}_2 . In addition, there are ϵ -transitions to states which generate w_{2i} and stop (this is in case the computation \hat{w} has an odd number of steps). A similar automaton \mathcal{A}_3 can be constructed which ensures that the configuration w_{2i+2} occurring at time $2i+2$ follows from the configuration w_{2i+1} occurring at time $2i+1$ for $i \geq 0$. Thus, a timed word \hat{w} represents a valid configuration of \mathcal{T} iff it belongs to $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2) \cap \mathcal{L}(\mathcal{A}_3)$. The result follows by noting that \mathcal{T} accepts a string iff $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2) \cap \mathcal{L}(\mathcal{A}_3) \neq \emptyset$. \square